M.SC. MATHEMATICS

MAL-643

Mechanics of Solids-II

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CONTENTS

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Chapter - 1

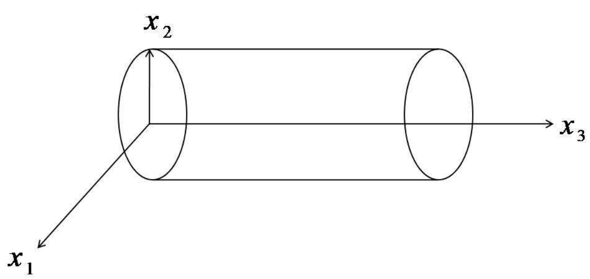
Two-Dimensional elastostatic Problems

1.1 Objectives

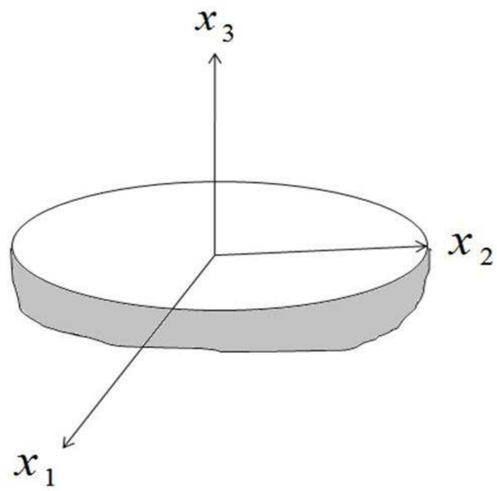
In this Chapter, we familiarize the students with Two-dimensional elastostatic problems. We shall discuss about some basic definitions of Plane strain deformation, Principal strains and directions for plane strain deformation, Anti-plane strain, Plane stress deformation, Generalized plane stress and Airy stress function. Examples are also given to illustrate these topics.

1.2 Introduction

The two-dimensional problems with which we shall be concerned in this chapter fall into two physically distinct types: Plane strain deformation and Plane stress deformation. First of these problem arise in the study of deformation of large cylindrical bodies acted upon by the external forces so distributed that the components of deformation in the direction of axis of the cylinder vanish and the remaining components do not vary along the length of the cylinder. This is the class of problems in plane strain deformation.



The other type appears in the study of the deformation of thin plates, the state of stress is characterized by the vanishing of the stress components in the direction of the thickness of the plate. These are the problems in plane stress. $\tau_{13}, \tau_{23}, \tau_{33}$ stress components are zero. $\tau_{11}, \tau_{22}, \tau_{12}$ are independent of x_3.



1.3 Plane strain deformation

A body is said to be in the state of plane strain (or deformation), parallel to the x_1x_2 -plane, if the displacement component u_3 vanishes identically and the other two displacement components u_1 and u_2 are functions of x_1 and x_2 coordinates only and independent of x_3 coordinate.

Thus, the state of plane strain deformation (parallel to x_1x_2 -plane) is characterised by the displacement components of the following type:

$$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2), \quad u_3 = 0 \tag{1}$$

Here strain components $e_{13} = e_{23} = e_{33} = 0$ and e_{11}, e_{22}, e_{12} are independent of x_3, i.e., non vanishing components of strain are

$$e_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) \quad ; \quad \alpha, \beta = 1,\ 2$$

Also non vanishing components of rotation tensor are

$$w_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} - u_{\beta,\alpha}\right) \quad ; \quad \alpha, \beta = 1,\ 2$$

The stresses follow from stress-strain relations

$\tau_{ij} = \lambda \delta_{ij}\ \vartheta + 2\mu e_{ij}$, $\qquad \vartheta = e_{ii}$ where λ and μ are Lame's constants and δ_{ij} is Kronecker delta.

i.e., the stresses are given by

$$\tau_{\alpha\beta} = \lambda \delta_{\alpha\beta}\, \vartheta + \mu\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) \quad ; \quad \alpha, \beta = 1,2 \qquad (2)$$

where $\vartheta = e_{11} + e_{22} = \left(u_{1,1} + u_{2,2}\right)$

From here, we get

$$\tau_{13} = \tau_{23} = 0, \quad \tau_{33} = \lambda \vartheta$$

From (2), we get

$$\tau_{11} + \tau_{22} = 2(\lambda + \mu)(e_{11} + e_{22})$$
$$\Rightarrow (e_{11} + e_{22}) = \frac{\tau_{11} + \tau_{22}}{2(\lambda + \mu)}$$

So

$$\tau_{33} = \lambda(e_{11} + e_{22}) = \frac{\lambda(\tau_{11} + \tau_{22})}{2(\lambda + \mu)}$$

$\Rightarrow \tau_{33} = \sigma(\tau_{11} + \tau_{22})$, where $\sigma = \dfrac{\lambda}{2(\lambda+\mu)}$ is Poisson's ratio.

$\Rightarrow \tau_{33}$ is expressed in terms of τ_{11} and τ_{22} .

From equilibrium equations

$\tau_{ij,j} = -F_i$

The components F_1 and F_2 of body forces must be independent of x_3 so far as τ_{ij} do not depend on x_3. Also $F_3 = 0$, since τ_{33} is not a function of x_3.

Here equilibrium equations become

$$\tau_{\alpha\beta,\beta} = -F_\alpha(x_1, x_2) \tag{3}$$

$$\Rightarrow \tau_{11,1} + \tau_{12,2} = -F_1(x_1, x_2) \qquad (\because \tau_{13} = 0) \tag{4}$$

$$\tau_{21,1} + \tau_{22,2} = -F_2(x_1, x_2) \qquad (\because \tau_{23} = 0) \tag{5}$$

Substitute (2) into (3), we get Navier's equations

Equation (2) is

$$\tau_{\alpha\beta} = \lambda\delta_{\alpha\beta}\vartheta + \mu(u_{\alpha,\beta} + u_{\beta,\alpha})$$

$$\Rightarrow \tau_{\alpha\beta,\beta} = \lambda\delta_{\alpha\beta}\frac{\partial\vartheta}{\partial\beta} + \mu(u_{\alpha,\beta\beta} + u_{\beta,\alpha\beta})$$

$$\Rightarrow \tau_{\alpha\beta,\beta} = \lambda\delta_{\alpha\beta}\vartheta_{,\beta} + \mu(\nabla^2 u_\alpha + u_{\beta,\beta\alpha})$$
$$= \lambda\vartheta_{,\alpha} + \mu(\nabla^2 u_\alpha + \vartheta_{,\alpha})$$

where $\vartheta = u_{\beta,\beta}$

Hence equation (3) $\Rightarrow -F_\alpha = (\lambda+\mu)\vartheta_{,\alpha} + \mu\nabla^2 u_\alpha$

or $$(\lambda+\mu)\vartheta_{,\alpha}+\mu\nabla^2 u_\alpha=-F_\alpha(x_1,x_2) \tag{6}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2} \quad .$$

(6) are known as Navier's equations.

We know that strain-stress relations are given by

$$e_{ij}=\frac{1+\sigma}{E}\tau_{ij}-\frac{\sigma}{E}\theta$$

where

$$\theta=(\tau_{11}+\tau_{22}+\tau_{33})$$

Therefore

$$\begin{aligned} e_{11} &= \frac{1+\sigma}{E}\tau_{11}-\frac{\sigma}{E}(\tau_{11}+\tau_{22}+\tau_{33}) \\ &= \frac{1+\sigma}{E}\tau_{11}-\frac{\sigma}{E}(\tau_{11}+\tau_{22}+\sigma(\tau_{11}+\tau_{22})) \\ &= \frac{1+\sigma}{E}\tau_{11}-\frac{\sigma}{E}(1+\sigma)(\tau_{11}+\tau_{22}) \end{aligned}$$

or

$$u_{1,1}=e_{11}=\frac{1+\sigma}{E}[(1-\sigma)\tau_{11}-\sigma\tau_{22}] \ , \tag{7a}$$

$$u_{2,2}=e_{22}=\frac{1+\sigma}{E}[(1-\sigma)\tau_{22}-\sigma\tau_{11}] \tag{7b}$$

and

$$2e_{12} = \left(u_{1,2} + u_{2,1}\right) = \frac{2(1+\sigma)\tau_{12}}{E} \tag{7c}$$

Five out of six compatibility equations are identically satisfied.

The only compatibility equation to be considered is

$$e_{11,22} + e_{22,11} = 2e_{12,12}$$

Using (7), we get

$$(1-\sigma)\left(\tau_{11,22} + \tau_{22,11}\right) - \sigma\left(\tau_{11,11} + \tau_{22,22}\right) = 2\tau_{12,12} \tag{8}$$

Differentiate (4) w.r.t. x_1 and (5) w. r. t. x_2 and adding, we get

$$\tau_{11,11} + \tau_{22,22} + 2\tau_{12,12} + F_{1,1} + F_{2,2} = 0 \tag{9}$$

From (8) and (9), we get

$$\tau_{11,11} + \tau_{22,22} + (1-\sigma)\left(\tau_{11,22} + \tau_{22,11}\right) - \sigma\left(\tau_{11,11} + \tau_{22,22}\right) + F_{1,1} + F_{2,2} = 0$$

$$\Rightarrow \quad (1-\sigma)\left[\tau_{11,11} + \tau_{22,22} + \tau_{11,22} + \tau_{22,11}\right] + F_{1,1} + F_{2,2} = 0$$

$$\Rightarrow \quad (1-\sigma)\left[\nabla^2\left(\tau_{11} + \tau_{22}\right)\right] + F_{1,1} + F_{2,2} = 0$$

$$\Rightarrow \quad \nabla^2\left(\tau_{11} + \tau_{22}\right) + \frac{F_{1,1} + F_{2,2}}{(1-\sigma)} = 0$$

Since $\sigma = \dfrac{\lambda}{2(\lambda+\mu)} \quad \Rightarrow \quad \dfrac{1}{1-\sigma} = \dfrac{2(\lambda+\mu)}{\lambda+2\mu}$

So, we get

$$\nabla^2\left(\tau_{11} + \tau_{22}\right) + \frac{2(\lambda+\mu)}{\lambda+2\mu}\left(F_{1,1} + F_{2,2}\right) = 0$$

$$\Rightarrow \quad \nabla^2\theta_1 = -\frac{2(\lambda+\mu)}{\lambda+2\mu}\left(F_{1,1} + F_{2,2}\right) \tag{10}$$

where $\theta_1 = (\tau_{11} + \tau_{22})$

Equation (10) is the compatibility equation in terms of stresses.

In cylindrical coordinates (r, θ, z) :

If u_r, u_θ, u_z are displacement components, then the strains in terms of displacements relations are given by

$$e_{rr} = \frac{\partial u_r}{\partial r} = u_{r,r}$$

$$e_{\theta\theta} = \frac{1}{r}\left[\frac{\partial u_\theta}{\partial \theta} + u_r\right]$$

$$e_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right]$$

$$e_{rz} = e_{\theta z} = e_{zz} = 0$$

where displacement components under plane strain conditions are given by

$$u_r = u_r(r, \theta), \quad u_\theta = u_\theta(r, \theta), \quad u_z = 0$$

Strain-stress relations are:

$$e_{rr} = \frac{1+\sigma}{E}[(1-\sigma)\tau_{rr} - \sigma\tau_{\theta\theta}]$$

and $e_{\theta\theta} = \frac{1+\sigma}{E}[(1-\sigma)\tau_{\theta\theta} - \sigma\tau_{rr}]$

$$e_{r\theta} = \frac{(1+\sigma)\tau_{r\theta}}{E}$$

$$\tau_{11} + \tau_{22} = \tau_{rr} + \tau_{\theta\theta}$$

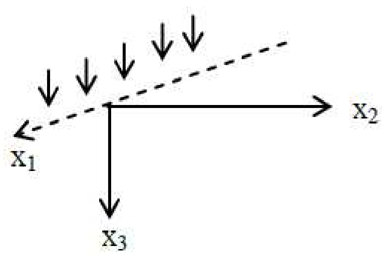
$$F_{1,1} + F_{2,2} = div\vec{F} = F_{r,r} + \frac{F_{\theta,\theta}}{r} + \frac{F_r}{r}$$

Compatibility equations:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial \theta^2}\right)(\tau_{rr} + \tau_{\theta\theta}) + \frac{1}{1-\sigma}\left(F_{r,r} + \frac{F_{\theta,\theta}}{r} + \frac{F_r}{r}\right) = 0$$

Examples of Plane strain deformations

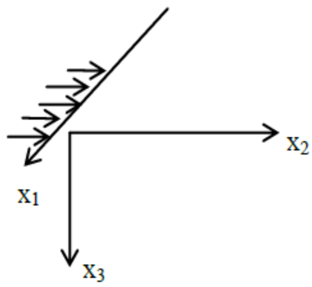
(A) The problem of stresses in an elastic semi-infinite medium subjected to a vertical line-load is a plane strain problem.



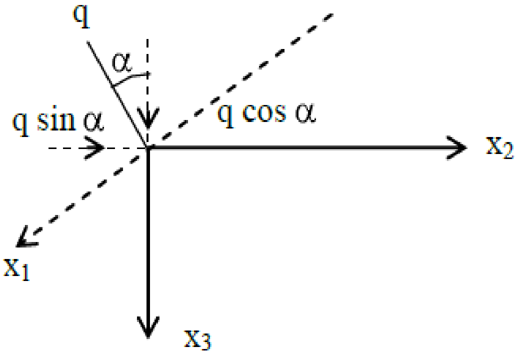
Here, the line-load extends to infinity on both sides of the origin. The displacement components are of the type

$$u_1 = 0, \quad u_2 = u_2(x_2, x_3), \qquad u_3 = u_3(x_2, x_3)$$

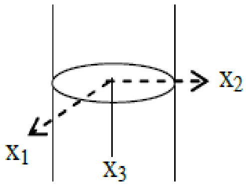
(B) The problem of determination of stresses resulting from a tangential line-load at the surface of a semi-infinite medium is a plane strain problem.



(C) The stresses and displacements in a semi-infinite elastic medium subjected to inclined loads can be obtained by superposition of the vertical and horizontal cases. If the components of the line-load are $q\cos\alpha$ and $q\sin\alpha$, the stresses can be determined.



(D) The problem of deformation of an infinite cylinder by a force in the x_1x_2 - plane is a plane strain problem.



In Cartesian coordinates

$u_1 = u_1(x_1, x_2), \quad u_2 = u_2(x_1, x_2), \quad u_3 = 0.$

In cylindrical coordinates

$u_r = u(r, \theta), \quad u_\theta = v(r, \theta), \quad u_z = 0$

1.4 Principal Strains and Directions for Plane Strain Deformation

A deformation for which the strain components e_{11}, e_{22} and e_{12} are independent of x_3 and $e_{13} = e_{23} = e_{33} = 0$ is called a plane strain deformation parallel to the $x_1 x_2$-plane.

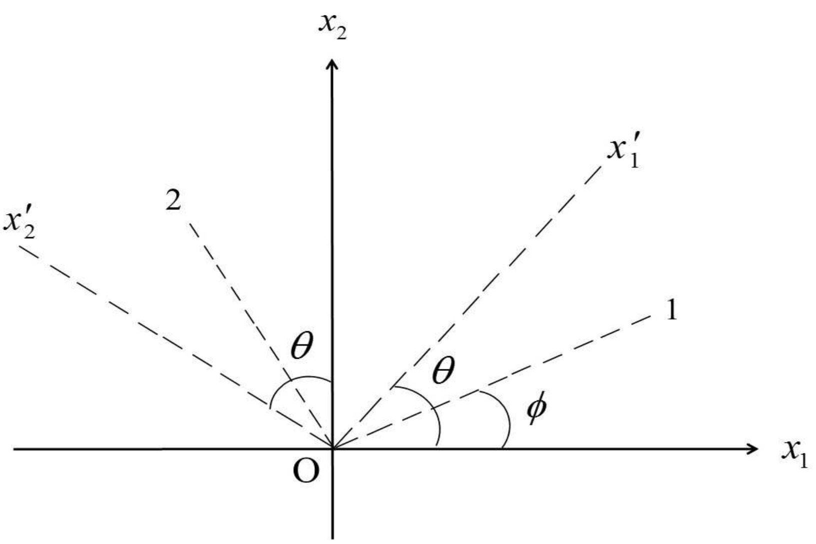
For such a deformation, the principal strain in the direction of x_3-axis is zero and the strain quadric of Cauchy

$$e_{ij}\, x_i\, x_j = \pm k^2 \quad , \qquad (1)$$

becomes

$$e_{11}\, x_1^2 + 2e_{12}\, x_1 x_2 + e_{22}\, x_2^2 = \pm k^2 \quad , \qquad (2)$$

which represents a cylinder in three-dimensions. Let the axes be rotated about x_3-axis through an angle θ to get new axes $O\, x_1'\, x_2'\, x_3'$.



Let $a_{ij} = \cos(x_i', x_j)$ (3)

Then

	x_1	x_2	x_3
x_1'	$\cos\theta$	$\sin\theta$	0
x_2'	$-\sin\theta$	$\cos\theta$	0
x_3'	0	0	1

The strains e'_{pq} relative to primed system are given by the law

$$e'_{pq} = a_{pi}\, a_{qj}\, e_{ij} \tag{5}$$

For (ij) = (11), (22), (12), (21), we find

$$
\begin{aligned}
e'_{11} &= a_{1i}\, a_{1j}\, e_{ij} \\
&= a_{11}^2\, e_{11} + a_{12}^2\, e_{22} + a_{11}\, a_{12} e_{12} + a_{12}\, a_{11} e_{12} \\
&= (\cos^2\theta)\, e_{11} + (\sin^2\theta)\, e_{22} + 2\sin\theta\, \cos\theta\, e_{12} \\
&= e_{11}\left(\frac{1+\cos 2\theta}{2}\right) + e_{22}\left(\frac{1-\cos 2\theta}{2}\right) + e_{12}\, \sin 2\theta \\
&= \frac{1}{2}(e_{11}+e_{22}) + \frac{1}{2}(e_{11}-e_{22})\cos 2\theta + e_{12}\sin 2\theta
\end{aligned}
$$

$$
e'_{11} = \frac{1}{2}(e_{11}+e_{22}) + \frac{1}{2}(e_{11}-e_{22})\cos 2\theta + e_{12}\sin 2\theta \tag{6a}
$$

Similarly

$$
e'_{22} = \frac{1}{2}(e_{11}+e_{22}) - \frac{1}{2}(e_{11}-e_{22})\cos 2\theta - e_{12}\sin 2\theta \tag{6b}
$$

$$
e'_{12} = -\frac{1}{2}(e_{11}-e_{22})\sin 2\theta + e_{12}\cos 2\theta \tag{6c}
$$

$$
e'_{31} = e'_{32} = e'_{33} = 0 \tag{6d}
$$

The principal directions in the $x_1 x_2$ -plane are given by

$$
e'_{12} = 0
$$

This gives

$$
\frac{\sin 2\theta}{e_{12}} = \frac{\cos 2\theta}{\frac{1}{2}(e_{11}-e_{22})} = \frac{1}{\sqrt{e_{12}^2 + \frac{1}{4}(e_{11}-e_{22})^2}} \tag{7a}
$$

and

$$
\tan 2\theta = \frac{e_{12}}{\frac{1}{2}(e_{11}-e_{22})} = \frac{2e_{12}}{(e_{11}-e_{22})} \tag{7b}
$$

Let ϕ be the angle which the principal directions O_1 and O_2 make with the old axes in the x_1x_2-plane. Then

$$\tan 2\phi = \frac{2e_{12}}{(e_{11} - e_{22})} \tag{8}$$

The principal strains e_1 and e_2 given by equations (6a, b) and (7a). We find $\left(e_1 = e_{11}^1 \ , \ e_2 = e_{22}^1\right)$

$$e_1 \ , e_2 = \frac{1}{2}(e_{11} + e_{22}) \pm \sqrt{\frac{1}{4}(e_{11} - e_{22})^2 + e_{12}^2} \tag{9}$$

the shearing strain e'_{12} will be maximum when

$$\frac{d}{d\theta} e'_{12} = 0$$

$$\Rightarrow \quad -(e_{11} - e_{22}) \cos 2\theta - 2e_{12} \sin 2\theta = 0$$

$$\Rightarrow \quad \frac{\cos 2\theta}{e_{12}} = \frac{\sin 2\theta}{-\frac{1}{2}(e_{11} - e_{22})} = \frac{1}{\sqrt{e_{12}^2 + \frac{1}{4}(e_{11} - e_{22})^2}} \tag{10a}$$

This gives the direction in which the shearing strain e'_{12} is maximum and maximum value of e'_{12} is given by equations (6c) and (10a). We find

$$e'_{12}\ {}_{\max} = \sqrt{e_{12}^2 + \frac{1}{4}(e_{11} - e_{22})^2} \tag{10b}$$

From equations (9) and (10b), we obtain

$$\frac{e_1 - e_2}{2} = e'_{12}\ {}_{\max} \tag{11}$$

This shows that maximum value of shearing strain is half of the difference of two principal strains in the x_1x_2 plane.

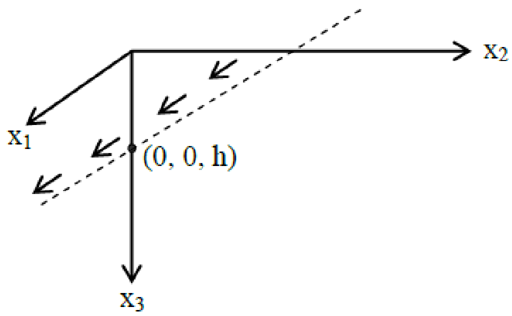
1.5 Anti-plane strain

A body is said to be in the state of anti-plane deformation parallel to x_1x_2-plane if

$u_1 = u_2 = 0,\ u_3 = u_3(x_1, x_2)$.

Example of Anti-plane Deformation

Suppose that a force is applied along the line which is parallel to x_1-axis and is situated at a depth h below the free-surface of an elastic isotropic half-space.



The resulting deformation is that of anti-plane strain deformation with

$u_1 = u_1(x_2, x_3),\ \ u_2 = u_3 = 0$

Remark: - Two-dimensional problems in acoustics are antiplane strain problems.

1.6 Plane stress deformation

An elastic body is said to be in the state of plane stress deformation parallel to the x_1x_2-plane, if stress components $\tau_{13} = \tau_{23} = \tau_{33} = 0$ and $\tau_{11}, \tau_{22}, \tau_{12}$ are independent of x_3.

From stress-strain relations,

$$\tau_{ij} = \lambda\delta_{ij}\vartheta + 2\mu e_{ij}, \qquad \vartheta = e_{ii} = u_{i,i} \tag{1}$$

$$\tau_{ij} = \lambda\delta_{ij}\vartheta + 2\mu\left(u_{i,j} + u_{j,i}\right)$$

$$\Rightarrow \qquad \tau_{33} = \lambda(e_{11} + e_{22} + e_{33}) + 2\mu e_{33}$$

$$\Rightarrow \qquad 0 = (\lambda + 2\mu)e_{33} + \lambda(e_{11} + e_{22})$$

$$\Rightarrow \qquad u_{3,3} = e_{33} = \frac{-\lambda(e_{11} + e_{22})}{\lambda + 2\mu} = \frac{-\lambda(u_{1,1} + u_{2,2})}{\lambda + 2\mu}$$

Strain component e_{33} is not independent but it depends on e_{11}, e_{22}, i.e., $e_{33} \neq 0$.

By definition of plane stress, $\tau_{13} = \tau_{23} = 0$ and non-zero stress components are $\tau_{11}, \tau_{22}, \tau_{12}$.

From (1), we have $\tau_{11} = \lambda\vartheta + 2\mu e_{11} = \lambda(e_{11} + e_{22} + e_{33}) + 2\mu e_{11}$

$$\Rightarrow \qquad \tau_{11} = \frac{2\lambda\mu}{\lambda + 2\mu}(e_{11} + e_{22}) + 2\mu e_{11} \text{ (using value of } e_{33}\text{)}$$

and

$$\tau_{22} = \frac{2\lambda\mu}{\lambda + 2\mu}(e_{11} + e_{22}) + 2\mu e_{22}$$

$$\tau_{12} = 2\mu e_{12} = \mu(u_{1,2} + u_{2,1})$$

Combining these equations, we get

$$\tau_{\alpha\beta} = \frac{2\lambda\mu}{\lambda + 2\mu}\delta_{\alpha\beta}\vartheta_1 + \mu\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) \tag{2}$$

where $\vartheta_1 = \left(e_{11} + e_{22}\right) = \left(u_{1,1} + u_{2,2}\right)$

Also

$$e_{kk} = \left(e_{11} + e_{22} + e_{33}\right) = \frac{2\mu}{\lambda + 2\mu}\left(e_{11} + e_{22}\right) \tag{3}$$

and from stress-strain relations,

$$e_{ij} = \frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\theta\delta_{ij},$$

We have

$$e_{11} = \frac{1+\sigma}{E}\tau_{11} - \frac{\sigma}{E}\left(\tau_{11} + \tau_{22}\right) = \frac{1}{E}\left[\tau_{11} - \sigma\tau_{22}\right] \tag{4}$$

and $e_{22} = \dfrac{1}{E}\left[\tau_{22} - \sigma\tau_{11}\right]$ (5)

$$e_{12} = \frac{1+\sigma}{E}\tau_{12} \tag{6}$$

$$e_{33} = \frac{-\sigma\left(\tau_{11} + \tau_{22}\right)}{E}, \quad e_{13} = e_{23} = 0$$

Using equation (2), the Equilibrium equations becomes

$$\tau_{\alpha\beta,\beta} + F_\alpha = 0, \quad \alpha, \beta = 1, 2$$

$$\Rightarrow \left[\frac{2\lambda\mu}{\lambda + 2\mu}\delta_{\alpha\beta}\vartheta_1 + \mu\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right)\right]_{,\beta} + F_\alpha = 0$$

$$\Rightarrow \left[\frac{2\lambda\mu}{\lambda+2\mu} \delta_{\alpha\beta} \frac{\partial \vartheta_1}{\partial x_\beta} + \mu\left(u_{\alpha,\beta\beta} + u_{\beta,\alpha\beta}\right)\right] + F_\alpha = 0$$

$$\Rightarrow \left[\frac{2\lambda\mu}{\lambda+2\mu} \frac{\partial \vartheta_1}{\partial x_\alpha} + \mu\left(\nabla_1^2 u_\alpha + u_{\beta,\beta\alpha}\right)\right] + F_\alpha = 0$$

$$\Rightarrow \left(\frac{2\lambda\mu}{\lambda+2\mu} + \mu \right) \frac{\partial \vartheta_1}{\partial x_\alpha} + \mu \nabla_1^2 u_\alpha = -F_\alpha$$

where

$$\vartheta_1 = (e_{11} + e_{22}) = (u_{1,1} + u_{2,2}) \quad \text{and} \quad \nabla_1{}^2 \equiv \frac{\partial^2}{\partial x_1{}^2} + \frac{\partial^2}{\partial x_2{}^2}.$$

If we put $\bar{\lambda} = \dfrac{2\lambda\mu}{\lambda+2\mu}$

we get

$$\left(\bar{\lambda} + \mu\right) \frac{\partial \vartheta_1}{\partial x_\alpha} + \mu \nabla_1^2 u_\alpha = -F_\alpha$$

and $\tau_{\alpha\beta} = \left[\bar{\lambda}\, \delta_{\alpha\beta} \vartheta_1 + \mu\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right)\right]$

$$\nabla^2(\tau_{11} + \tau_{22}) + (1+\sigma)\left(F_{1,1} + F_{2,2}\right) = 0$$

The only compatibility equation to be satisfied is

$$\left(e_{11,22} + e_{22,11}\right) = 2e_{12,12}$$

$$\Rightarrow \left(\tau_{11,22} + \tau_{22,11}\right) - \sigma\left(\tau_{11,11} + \tau_{22,22}\right) = 2(1+\sigma)\tau_{12,12} \qquad (7)$$

Here equilibrium equations become

$$\tau_{\alpha\beta,\beta} = -F_\alpha(x_1, x_2)$$

$$\Rightarrow \tau_{11,1} + \tau_{12,2} = -F_1(x_1, x_2) \quad (8)$$

$$\tau_{21,1} + \tau_{22,2} = -F_2(x_1, x_2) \quad (9)$$

Differentiate (8) w.r.t x_1 and (9) w.r.t. x_2 and adding, we get

$$\tau_{11,11} + \tau_{22,22} + 2\tau_{12,12} + F_{1,1} + F_{2,2} = 0$$

$$\tau_{11,11} + \tau_{22,22} + \frac{1}{(1+\sigma)}\left(\tau_{11,22} + \tau_{22,11} - \sigma(\tau_{11,11} + \tau_{22,22})\right) + F_{1,1} + F_{2,2} = 0$$

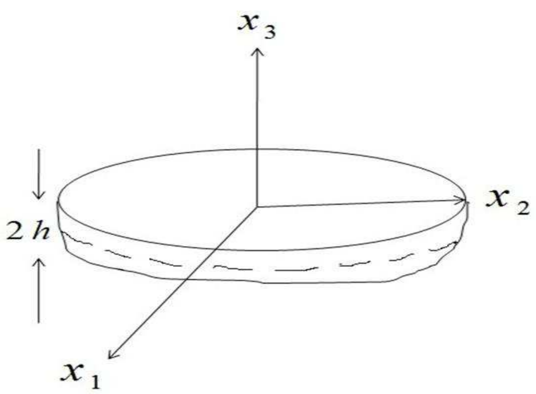
$$\Rightarrow [\tau_{11,11} + \tau_{22,22} + \tau_{11,22} + \tau_{22,11}] + (1+\sigma)(F_{1,1} + F_{2,2}) = 0$$

$$\Rightarrow [\nabla^2(\tau_{11} + \tau_{22})] + (1+\sigma)(F_{1,1} + F_{2,2}) = 0$$

which is required compatible equation.

1.7 GENERALIZED PLANE STRESS

Consider a thin flat plate of thickness 2h. We take the middle plane of the plate as $x_3 = 0$ plane so that the two faces of the plate are $x_3 = h$ and $x_3 = -h$.



We make the following assumptions:

(a) The faces of plate are free from applied loads.

(b) The surface forces acting on the edge (curved surface) of the plate lie in planes parallel to the middle plane $(x_3 = 0)$, i.e., parallel to x_1x_2 -plane and are symmetrically distributed w.r.t the middle plane $x_3 = 0$.

(c) $F_3 = 0$ and components F_1 and F_2 of the body force are symmetrically distributed w.r.t the middle plane.

Under these assumptions, the points of the middle plane will not undergo any deformation in the x_3-direction.

Here $\bar{u}_1 = \bar{u}_1(x_1, x_2)$, $\bar{u}_2 = \bar{u}_2(x_1, x_2)$, $\bar{u}_3 = 0$

$\bar{\tau}_{13} = \bar{\tau}_{23} = \bar{\tau}_{33} = 0$

$$\bar{\tau}_{11} = (\bar{\lambda} + 2\mu)\bar{e}_{11} + \bar{\lambda}\bar{e}_{22}$$

$$\bar{\tau}_{22} = (\bar{\lambda} + 2\mu)\,\bar{e}_{22} + \bar{\lambda}\bar{e}_{11}$$

$$\bar{\tau}_{12} = 2\mu\,\bar{e}_{12}$$

$$\bar{e}_{33} = \frac{-\lambda}{\lambda + 2\mu}(\bar{e}_{11} + \bar{e}_{22}) \qquad \text{where} \quad \bar{\lambda} = \frac{2\lambda\mu}{\lambda + 2\mu}$$

$$\bar{\tau}_{\alpha\beta} = \bar{\lambda}\,\delta_{\alpha\beta}\bar{\vartheta}_1 + \mu(\bar{u}_{\alpha,\beta} + \bar{u}_{\beta,\alpha})$$

where $\bar{\vartheta}_1 = \bar{u}_{\alpha,\alpha}$

Also $e_{kk} = (e_{11} + e_{22} + e_{33}) = \dfrac{2\mu}{\lambda + 2\mu}(e_{11} + e_{22})$

The Navier's equations are given by

$$\left(\bar{\lambda}+\mu\right)\frac{\partial \bar{\vartheta}_1}{\partial x_\alpha}+\mu\nabla^2\bar{u}_\alpha = -\bar{F}_\alpha\left(x_1, x_2\right)$$

and

$$\nabla^2\bar{\vartheta}_1 = -\frac{2\left(\bar{\lambda}+\mu\right)}{\left(\bar{\lambda}+2\mu\right)}\bar{F}_{\alpha,\alpha}$$

which is compatibility equation, where $\bar{\vartheta}_1 = \bar{\tau}_{11}+\bar{\tau}_{22}$

1.8 Airy Stress Function

Considering Plane strain case

$$u_1 = u_1\left(x_1, x_2\right), \quad u_2 = u_2\left(x_1, x_2\right) \tag{1}$$

$$e_{11} = \frac{\partial u_1}{\partial x_1}, \quad e_{22} = \frac{\partial u_2}{\partial x_2} \tag{2}$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\right) \tag{3}$$

$$\tau_{33} = \lambda\left(e_{11}+e_{22}\right) \tag{4}$$

$$\tau_{11} = \left(\lambda+2\mu\right)e_{11}+\lambda e_{22} \tag{5}$$

$$\tau_{22} = \left(\lambda+2\mu\right)e_{22}+\lambda e_{11} \tag{6}$$

$$\tau_{12} = 2\mu e_{12} \tag{7}$$

Here equilibrium equations in the absence of body forces are $\tau_{\alpha\beta,\beta} = 0$

$$\Rightarrow \tau_{11,1}+\tau_{12,2} = 0 \tag{8}$$

$$\tau_{21,1}+\tau_{22,2} = 0 \tag{9}$$

Therefore, there exists an Airy's stress function $\phi\left(x_1, x_2\right)$

s. t. $\tau_{11}=\dfrac{\partial^2\phi}{\partial x_2^2},\qquad \tau_{22}=\dfrac{\partial^2\phi}{\partial x_1^2},\qquad \tau_{12}=\dfrac{-\partial^2\phi}{\partial x_1\partial x_2}$ (10)

using (10), (8) and (9) are identically satisfied. The compatibility equation is

$$\frac{\partial^2 e_{11}}{\partial x_2^2}+\frac{\partial^2 e_{22}}{\partial x_1^2}=2\frac{\partial^2 e_{12}}{\partial x_1\partial x_2} \tag{11}$$

Solving (5), (6), (7) for strains, we obtain the strains

$$\begin{aligned}
e_{11}&=\frac{1}{4\mu(\lambda+\mu)}\left[(\lambda+2\mu)\tau_{11}-\lambda\tau_{22}\right]\\
e_{22}&=\frac{1}{4\mu(\lambda+\mu)}\left[(\lambda+2\mu)\tau_{22}-\lambda\tau_{11}\right]\\
e_{12}&=\frac{1}{2\mu}\tau_{12}
\end{aligned} \tag{12}$$

From (11) and (12), we obtain the compatibility equation in terms of stresses,

$$\frac{\partial^2\left[(\lambda+2\mu)\tau_{11}-\lambda\tau_{22}\right]}{\partial x_2^2}+\frac{\partial^2\left[(\lambda+2\mu)\tau_{22}-\lambda\tau_{11}\right]}{\partial x_1^2}=\frac{4(\lambda+\mu)\partial^2\tau_{12}}{\partial x_1\partial x_2} \tag{13}$$

Equation (10) and (13) give

$$\nabla^2\nabla^2\phi=0 \tag{14}$$

where $\nabla^2\equiv\dfrac{\partial^2}{\partial x_1{}^2}+\dfrac{\partial^2}{\partial x_2{}^2}$ (15)

or $\nabla^4\phi\equiv\phi_{,1111}+2\phi_{,1122}+\phi_{,2222}=0$

Here ϕ is biharmonic function in the absence of body forces.

From equation (2) to (5), we have stresses

$$\tau_{11} = \frac{\partial^2\phi}{\partial x_2^2} = (\lambda + 2\mu)\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2}$$

$$\tau_{22} = \frac{\partial^2\phi}{\partial x_1^2} = (\lambda + 2\mu)\frac{\partial u_2}{\partial x_2} + \lambda\frac{\partial u_1}{\partial x_1} \qquad (16)$$

$$\tau_{12} = \frac{-\partial^2\phi}{\partial x_1 \partial x_2} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$$

Solving first two equations of (16), we get

$$2\mu\frac{\partial u_1}{\partial x_1} = -\frac{\partial^2\phi}{\partial x_1^2} + \frac{1}{2\alpha}\nabla^2\phi$$

$$2\mu\frac{\partial u_2}{\partial x_2} = -\frac{\partial^2\phi}{\partial x_2^2} + \frac{1}{2\alpha}\nabla^2\phi \qquad (17)$$

where $\alpha = \frac{(\lambda + \mu)}{(\lambda + 2\mu)}$

Integrating (17), we get

$$2\mu u_1 = \frac{-\partial\phi}{\partial x_1} + \frac{1}{2\alpha}\int \nabla^2\phi \, dx_1 + f(x_2) \qquad (18)$$

$$2\mu u_2 = -\frac{\partial\phi}{\partial x_2} + \frac{1}{2\alpha}\int \nabla^2\phi \, dx_2 + g(x_1)$$

where $f(x_2)$ and $g(x_1)$ are arbitrary constants.

Due to 3rd equation of (16), we can neglect $f(x_2)$ and $g(x_1)$.

Then equation (18) becomes

$$2\mu u_1 = \frac{-\partial\phi}{\partial x_1} + \frac{1}{2\alpha}\int \nabla^2\phi \, dx_1 \qquad (19)$$

$$2\mu u_2 = -\frac{\partial \phi}{\partial x_2} + \frac{1}{2\alpha}\int \nabla^2 \phi \, dx_2$$

which gives displacement in case of plane strain.

Here $\frac{1}{2\alpha} = \frac{(\lambda + 2\mu)}{2(\lambda + \mu)} = (1-\sigma)$, for plane strain

and $\frac{1}{2\alpha} = (1+\sigma)^{-1}$ for plane stress.

1.9 Summary

In this chapter we have discussed about Plane strain, Principal strains and directions for plane strain deformation, Anti-plane strain and plane stress deformation and Airy stress function.

Keywords Plane strain, Anti-plane strain, Plane stress, generalized plane stress, Airy stress function, Biharmonic function.

1.10 Self-assessment Questions

Q 1. Discuss the principal stresses and principal directions of stress in a state of plane stress.

Q 2. What is plane deformation? Derive Beltrami-Michell compatibility equations for plane deformation.

Q 3. Describe physically the Plane stress problems and derive the relevant field equations.

Q 4. Obtain Navier's equations for the Plane strain and for Plane stress problems.

Q 5. What is Generalized plane stress? Derive the relevant field equations.

Q 6. Explain Airy's stress function.

1.11 Suggested Readings

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Chapter - 2

Displacements and stresses in terms of two analytic functions

2.1 Objectives

This chapter is concerned with the general method of solutions of two-dimensional boundary value problems in elasticity. In this Chapter, we familiarize the students with the General solution of Biharmonic equation. We derive the formula for stresses and displacements in terms of analytic functions. We shall discuss structure of functions and the arbitrariness in selection of functions $\phi(z)$ and $\psi(z)$ when the displacements or the stresses are given. We shall also discuss about first and second boundary value problems in plane elasticity.

2.2 Introduction

This chapter is devoted to a concise presentation of one general method of solution of certain broad classes of two-dimensional boundary-value problems in elasticity. The method is based on a reduction of the boundary-value problems in elasticity to the solutions of certain functional equations in a complex domain. The solution of the fundamental biharmonic boundary value problem can be made to depend on a certain general representation of the biharmonic function by means of two analytic functions of a complex variable.

2.3 General solution of the Biharmonic equation:

We find the solution in terms of two analytic functions. We consider the Biharmonic equation

$$\nabla^2\nabla^2\Phi = 0 \tag{1}$$

Let $\nabla^2\Phi = P_1(x_1, x_2)$, then $\nabla^2 P_1 = 0$

i.e., P_1 is harmonic function.

Let

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2$$
$$\Rightarrow x_1 = \frac{1}{2}(z + \bar{z}), \qquad x_2 = \frac{1}{2i}(z - \bar{z})$$

As P_1 is harmonic function in R (2D region), then there exists a conjugate harmonic function P_2 s.t.

$F(z) = P_1 + iP_2$ is analytic function in R.

Let

$$\psi_1(z) = \frac{1}{4}\int F(z)dz$$
$$\Rightarrow \psi_1(z) = \frac{1}{4}\int (P_1 + iP_2)\ dz = p_1 + ip_2 \tag{2}$$

where

$$p_1 = \frac{1}{4}\int P_1 dz, \quad p_2 = \frac{1}{4}\int P_2 dz$$

$\Rightarrow \psi_1(z)$ is also analytic function of z in R.

So, $\psi_1'(z) = \frac{1}{4}F(z) = \frac{1}{4}(P_1 + iP_2)$

Also from (2), $\psi_1'(z) = \left(\frac{\partial p_1}{\partial x_1} + i\frac{\partial p_2}{\partial x_1}\right)\frac{\partial z}{\partial x_1}$

$$\Rightarrow \psi_1'(z) = \left(\frac{\partial p_1}{\partial x_1} + i\frac{\partial p_2}{\partial x_1}\right) \qquad \left(\because \frac{\partial z}{\partial x_1} = 1\right)$$

$$\Rightarrow \frac{1}{4}(P_1 + iP_2) = \left(\frac{\partial p_1}{\partial x_1} + i\frac{\partial p_2}{\partial x_1}\right)$$

$$\Rightarrow \frac{1}{4}P_1 = \frac{\partial p_1}{\partial x_1} = p_{1,1}, \qquad \text{and} \qquad \frac{1}{4}P_2 = \frac{\partial p_2}{\partial x_1} = p_{2,1}$$

Using C-R equations, $p_{1,1} = p_{2,2} = \frac{1}{4}P_1$, $p_{2,1} = -p_{1,2} = \frac{1}{4}P_2$ (3)

Consider

$$\nabla^2[\Phi - p_1x_1 - p_2x_2] = P_1 - 2p_{1,1} - 2p_{2,2}$$

$$= P_1 - \frac{1}{2}P_1 - \frac{1}{2}P_1 = 0$$

$\left(\because \nabla^2 p_1x_1 = 2p_{1,1}, \nabla^2 p_2x_2 = 2p_{2,2}\right)$

$\Rightarrow \Phi - p_1x_1 - p_2x_2$ is harmonic function. Hence Φ has the structure as

$$\Phi - p_1x_1 - p_2x_2 = q_1(x_1, x_2) \tag{4}$$

where $q_1(x_1, x_2)$ is also harmonic function in R.

Because q_1 is harmonic function in R, then there exists a function q_2

s.t. $q_1 + iq_2 = \psi_2(z)$ is analytic.

So,

$\Phi = p_1x_1 + p_2x_2 + q_1(x_1, x_2)$ = Real $[\bar{z}\psi_1(z) + \psi_2(z)]$

If we denote the conjugate complex by bars, then

$$2\Phi = [\bar{z}\,\psi_1(z) + z\,\overline{\psi_1}(z) + \psi_2(z) + \overline{\psi_2}(z)] \tag{5}$$

where $\psi_1(z)$ and $\psi_2(z)$ are arbitrary analytic functions of x_1 and x_2.

2.4 Formula for stresses in terms of analytic functions:

The components $\tau_{\alpha\beta}$ of the stress tensor can be expressed in terms of the functions $\phi(z)$ and $\psi(z)$.

We denote

$$\psi_1(z) = \phi(z), \quad \psi_2(z) = \psi(z)$$

and Φ by U, which is stress function, then (5) of last article can be written as

$$2U = [\bar{z}\,\phi(z) + z\,\overline{\phi}(z) + \psi(z) + \overline{\psi}(z)] \tag{1}$$

The stresses in terms of Airy's stress function are

$$\tau_{11} = U_{,22} \quad , \quad \tau_{22} = U_{,11} \quad , \quad \tau_{12} = -U_{,12} \tag{2}$$

and

$$\begin{aligned} \tau_{11} + i\,\tau_{22} &= U_{,22} - iU_{,12} \\ &= -i\left(U_{,12} + iU_{,22}\right) \\ &= -i\frac{\partial}{\partial x_2}\left(U_{,1} + iU_{,2}\right) \end{aligned} \tag{3}$$

Similarly

$$\tau_{22} - i\,\tau_{12} = U_{,11} + iU_{,12} = \frac{\partial}{\partial x_1}\left(U_{,1} + iU_{,2}\right)$$

Let $z = x_1 + ix_2$, $\quad \bar{z} = x_1 - ix_2$

$$\Rightarrow x_1 = \frac{1}{2}(z + \bar{z}), \qquad x_2 = \frac{1}{2i}(z - \bar{z})$$

z and $\bar{z}$ are independent variables.

$$\frac{\partial}{\partial x_1} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial x_2} = i\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right) \tag{4}$$

$$\Rightarrow \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} = 2\frac{\partial}{\partial \bar{z}}$$

We calculate $(U,_1 + iU,_2)$

$$\frac{\partial U}{\partial x_1} + i\frac{\partial U}{\partial x_2} = 2\frac{\partial U}{\partial \bar{z}}$$

From (1), we have

$$2\frac{\partial U}{\partial \bar{z}} = \phi(z) + z\bar{\phi}'(z) + \bar{\psi}'(z)$$

Therefore, $\dfrac{\partial U}{\partial x_1} + i\dfrac{\partial U}{\partial x_2} = \phi(z) + z\bar{\phi}'(z) + \bar{\psi}'(z)$ (5)

From Eq. (3) and (5) on using (4) , we get

$$\tau_{11} + i\tau_{12} = -i^2\left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)\left[\phi(z) + z\bar{\phi}'(z) + \bar{\psi}'(z)\right]$$

$$= \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}}\right)\left[\phi(z) + z\bar{\phi}'(z) + \bar{\psi}'(z)\right]$$

$$\Rightarrow \tau_{11} + i\tau_{12} = \phi'(z) + \bar{\phi}'(z) - [z\phi''(z) + \bar{\psi}''(z)] \tag{6}$$

Similarly

$$\tau_{22} - i\tau_{12} = \left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}\right)\left[\phi(z) + z\bar{\phi}'(z) + \bar{\psi}'(z)\right]$$

$$= \phi'(z) + \bar{\phi}'(z) + [z\phi''(z) + \bar{\psi}''(z)] \qquad (7)$$

Adding (6) and (7), we get

$$\tau_{11} + \tau_{22} = 2\left[\phi'(z) + \bar{\phi}'(z)\right] = 4\,\text{Real}\ [\phi'(z)] \qquad (8)$$

Subtracting (6) from (7), we get

$$\tau_{22} - \tau_{11} - 2i\tau_{12} = 2\left[z\bar{\phi}''(z) + \bar{\psi}''(z)\right]$$

Taking its conjugate complex,

$$\tau_{22} - \tau_{11} + 2i\tau_{12} = 2\left[\bar{z}\phi''(z) + \psi''(z)\right] \qquad (9)$$

Adding these last two equations, we get

$$\tau_{22} - \tau_{11} = [z\bar{\phi}''(z) + \bar{\psi}''(z) + \bar{z}\phi''(z) + \psi''(z)] \qquad (10)$$

Equations (8) & (10) give stresses in terms of two analytic functions $\phi(z)$ & $\psi(z)$.

Further adding Eq. (8) and (10), we get

$$\Rightarrow 2\tau_{22} = 2[\phi'(z) + \bar{\phi}'(z)] + [z\bar{\phi}''(z) + \bar{\psi}''(z) + \bar{z}\phi''(z) + \psi''(z)]$$

$$\Rightarrow \tau_{22} = [\phi'(z) + \bar{\phi}'(z)] + \frac{1}{2}[z\bar{\phi}''(z) + \bar{\psi}''(z) + \bar{z}\phi''(z) + \psi''(z)]$$

Subtracting Eq. (10) from (8), we get

$$2\tau_{11} = 2[\phi'(z) + \bar{\phi}'(z)] - [z\bar{\phi}''(z) + \bar{\psi}''(z) + \bar{z}\phi''(z) + \psi''(z)]$$

or

$$\tau_{11} = [\phi'(z) + \bar{\phi}'(z)] - \frac{1}{2}[z\bar{\phi}''(z) + \bar{\psi}''(z) + \bar{z}\phi''(z) + \psi''(z)]$$

Therefore

$$\tau_{22} = 2 \text{ Real } \left[\phi'(z) + \frac{1}{2}\left\{\bar{z}\phi''(z) + \psi''(z)\right\}\right]$$

$$\tau_{11} = 2 \text{ Real } [\phi'(z)] - \text{Real}\left[\bar{z}\phi''(z) + \psi''(z)\right]$$

and

$$i\tau_{12} = \frac{1}{2}\left[\bar{z}\phi''(z) - z\bar{\phi}''(z) + \psi''(z) - \bar{\psi}''(z)\right] \qquad \text{(Using (9))}$$

2.5 Displacements in terms of two analytic functions:

For plane strain problems, the generalised Hooke's Law is given by

$$\tau_{\alpha\beta} = \lambda\vartheta_1\delta_{\alpha\beta} + \mu(u_{\alpha\beta} + u_{\beta\alpha})$$

$$\Rightarrow \tau_{11} = \lambda\vartheta_1 + 2\mu u_{1,1} = U_{,22} \qquad \text{(i)}$$

$$\tau_{22} = \lambda\vartheta_1 + 2\mu u_{2,2} = U_{,11} \qquad \text{(ii)} \qquad \text{(Using (2))}$$

$$\tau_{12} = \mu(u_{1,2} + u_{2,1}) = -U_{,12} \qquad \text{(iii)} \qquad (11)$$

Therefore

$$\tau_{11} + \tau_{22} = 2\lambda\vartheta_1 + 2\mu(u_{1,1} + u_{2,2})$$

$$\Rightarrow U_{,22} + U_{,11} = 2\lambda(u_{1,1} + u_{2,2}) + 2\mu(u_{1,1} + u_{2,2})$$

$$\Rightarrow U_{,22} + U_{,11} = 2(\lambda + \mu)\vartheta_1$$

$$\Rightarrow \vartheta_1 = \frac{1}{2(\lambda + \mu)}(U_{,22} + U_{,11}) \qquad (12)$$

From Eq. (12) and 11(i), we get

$$\frac{\lambda}{2(\lambda+\mu)}\left[U_{,22}+U_{,11}\right]+2\mu u_{1,1}=U_{,22}$$

$$\Rightarrow \frac{\lambda}{2(\lambda+\mu)}\nabla_1^2 U+2\mu u_{1,1}=U_{,22}+U_{,11}-U_{,11}=\nabla_1^2 U-U_{,11}$$

$$\Rightarrow 2\mu u_{1,1}=-U_{,11}+\left[1-\frac{\lambda}{2(\lambda+\mu)}\right]\nabla_1^2 U$$

$$\Rightarrow 2\mu u_{1,1}=-U_{,11}+\frac{\lambda+2\mu}{2(\lambda+\mu)}\nabla_1^2 U$$

Similarly

$$2\mu u_{2,2}=-U_{,22}+\frac{\lambda+2\mu}{2(\lambda+\mu)}\nabla_1^2 U \tag{13}$$

But we know that

$$\nabla_1^2 U=P_1 \Rightarrow P_1=4p_{1,1}=4p_{2,2}$$

Using it, equation (13) reduces to

$$2\mu u_{1,1}=-U_{,11}+\frac{2(\lambda+2\mu)}{(\lambda+\mu)}p_{1,1}$$

and

$$2\mu u_{2,2}=-U_{,22}+\frac{2(\lambda+2\mu)}{(\lambda+\mu)}p_{2,2}$$

Integrating above equations w.r.t. x_1 and x_2, respectively, we get

$$2\mu u_1 = -U_{,1} + \frac{2(\lambda+2\mu)}{(\lambda+\mu)} p_1 + f(x_2)$$

and (14)

$$2\mu u_2 = -U_{,2} + \frac{2(\lambda+2\mu)}{(\lambda+\mu)} p_2 + g(x_1)$$

where $f(x_2)$ and $g(x_1)$ are arbitrary constants.

Also from 11(iii), we have

$$\tau_{12} = -U_{,12} = \mu(u_{1,2} + u_{2,1})$$

Using (14), we get

$$-U_{,12} = \mu(u_{1,2} + u_{2,1}) = \left[-U_{,12} + \frac{(\lambda+2\mu)}{(\lambda+\mu)}(p_{1,2} + p_{2,1}) + \frac{f'(x_2) + g'(x_1)}{2}\right]$$

$$\Rightarrow 0 = \frac{f'(x_2) + g'(x_1)}{2}$$

$$\Rightarrow f'(x_2) = -g'(x_1) = \text{constant}, \ say = \alpha$$

Integrating, we get

$$f(x_2) = \alpha x_2 + \beta$$

$$g(x_1) = -\alpha x_1 + \gamma$$

where α, β, γ are constants.

So $f(x_2)$ and $g(x_1)$ correspond to rigid body displacements and can be neglected.

Then from (14), we have

$$2\mu u_1 = -U_{,1} + \frac{2(\lambda+2\mu)}{(\lambda+\mu)} p_1$$

and

$$2\mu u_2 = -U_{,2} + \frac{2(\lambda+2\mu)}{(\lambda+\mu)} p_2$$

$$2\mu\left(u_1+iu_2\right)=-\left(U_{,1}+iU_{,2}\right)+\frac{2\left(\lambda+2\mu\right)}{\left(\lambda+\mu\right)}\left(p_1+ip_2\right) \quad (15)$$

$2\mu\left(u_1+iu_2\right)=-\left[\phi(z)+z\overline{\phi}'(z)+\overline{\psi}'(z)\right]+\dfrac{2\left(\lambda+2\mu\right)}{\left(\lambda+\mu\right)}\phi(z)$ (on using (5) and Eq. (2) of previous article)

$$2\mu\left(u_1+iu_2\right)=\left[\kappa\phi(z)-z\overline{\phi}'(z)-\overline{\psi}'(z)\right], \quad (16)$$

where $\kappa=\dfrac{\left(\lambda+3\mu\right)}{\left(\lambda+\mu\right)}=3-4\sigma$, (17)

σ being the Poisson's ratio.

These are the expressions of displacements for plane strain problems.

It follows from equations (8), (9) and (16) that the components $\tau_{\alpha\beta}$ of the stress tensor and components u_α of the displacement vector are analytic functions of the real variables x_1 and x_2 throughout the interior of the region occupied by the body.

In the generalized plane stress problem, λ must be replaced by $\overline{\lambda}=\dfrac{2\lambda\mu}{\lambda+2\mu}$ and if the corresponding value of κ in (16) is denoted by

$$\overline{\kappa}=\frac{\left(\overline{\lambda}+3\mu\right)}{\left(\overline{\lambda}+\mu\right)}=\frac{5\lambda+6\mu}{3\lambda+2\mu}=\frac{3-\sigma}{1+\sigma}$$

2.6 The structure of functions $\phi(z)$ and $\psi(z)$:

Question. What is the difference in the forms of two sets of functions (ϕ,ψ) and (ϕ_0,ψ_0) that correspond to the same stress distribution in R?

Or discuss the arbitrariness in selection of functions $\phi(z)$ and $\psi(z)$ when the displacements or the stresses are given by

$$\tau_{11}+\tau_{22}=4\,\text{Real }[\phi'(z)]$$

$$\tau_{22}-\tau_{11}+2i\tau_{12}=2\left[\bar{z}\phi''(z)+\psi''(z)\right]$$

$$2\mu(u_1+iu_2)=\left[\kappa\phi(z)-z\overline{\phi}'(z)-\overline{\psi}'(z)\right]$$

Proof:

Case-I: - Let us consider two sets of functions (ϕ,ψ) and (ϕ_0,ψ_0) that correspond to the same stress distribution in R.

Then from relation

$$\tau_{11}+\tau_{22}=4\ \text{Real }[\phi'(z)]$$

We get

$$\text{Real }[\phi'(z)]=\text{Real }[\phi_0'(z)]$$

$\Rightarrow \phi'(z)$ and $\phi_0'(z)$ can differ only at the most by a complex quantity.

$\Rightarrow \phi_0'(z)=\phi'(z)+ic$, where c is real constant.

On integrating, we get

$$\phi_0(z)=\phi(z)+icz+\alpha \tag{1}$$

where α is any complex constant.

$\phi(z)$ and $\phi_0(z)$ can be replaced by $\phi(z)$ and $\phi(z)+icz+\alpha$ and these will give same stress distribution.

Also

$$\tau_{22} - \tau_{11} + 2i\tau_{12} = 2\left[\bar{z}\phi''(z) + \psi''(z)\right]$$

$$\Rightarrow \bar{z}\phi''(z) + \psi''(z) = \bar{z}\phi_0''(z) + \psi_0''(z)$$

Using (1), $\phi''(z) = \phi_0''(z)$,

then we have

$$\psi''(z) = \psi_0''(z)$$

On integrating twice, we get $\psi_0(z) = \psi(z) + \beta$

So, the state of stress in R will be unaltered if $\phi(z)$ is replaced by $\phi(z) + icz + \alpha$ and $\psi(z)$ by $\psi(z) + \beta$.

Case II- If displacement throughout R is satisfied.

Here

$$2\mu(u_1 + iu_2) = \left[\kappa\phi(z) - z\bar{\phi}'(z) - \bar{\psi}'(z)\right]$$

$$2\mu(u_1 + iu_2) = \left[\kappa\phi_0(z) - z\bar{\phi}_0'(z) - \bar{\psi}_0'(z)\right]$$

For same stress distribution, displacements are considered same.

$$\left[\kappa\phi(z) - z\bar{\phi}'(z) - \bar{\psi}'(z)\right] = \left[\kappa\phi_0(z) - z\bar{\phi}_0'(z) - \bar{\psi}_0'(z)\right]$$

From (1), Put $\phi_0(z) = \phi(z) + icz + \alpha$

$$\bar{\phi}_0'(z) = \bar{\phi}'(z) - ic$$

$$\Rightarrow \left[\kappa\phi(z) - z\overline{\phi}'(z) - \overline{\psi}'(z)\right] = \left[\kappa\phi(z) + icz\kappa + \kappa\alpha - z\overline{\phi}'(z) + icz - \overline{\psi}_0'(z)\right]$$

$$\Rightarrow \overline{\psi}_0'(z) - \overline{\psi}'(z) = icz(\kappa+1) + \kappa\alpha$$

Substitute $\psi_0(z) = \psi(z) + z\beta$ $\qquad (\because \psi_0'(z) = \psi'(z) + \beta)$

$$\overline{\psi}'(z) + \overline{\beta} - \overline{\psi}'(z) = icz(\kappa+1) + \kappa\alpha$$

$$\Rightarrow c = 0, \quad \overline{\beta} = \kappa\alpha$$

2.7 First and Second Boundary Value Problems

In this article it is shown that the fundamental boundary value problems in plane elasticity can be reduced to the determination of functions $\phi(z)$ and $\psi(z)$ from the prescribed values of certain combinations of these functions on the boundary of the region.

First B.V.P.:-

Stresses or loads are known on the boundary, i.e., stresses $(\tau_{11} = T_1, \tau_{22} = T_2)$ are known on the boundary, and $(\tau_{12} = -U_{,12})$

Then, we know that

$$U_{,1} = -\int T_2 ds = f_1(s) + \text{const.}$$

and

$$U_{,2} = -\int T_1 ds = f_2(s) + \text{const}.$$

$$\Rightarrow U_{,1} + iU_{,2} = f_1(s) + i\, f_2(s) + \text{const.}$$

$$\Rightarrow \phi(z) + z\overline{\phi}'(z) + \overline{\psi}'(z) = f_1(s) + if_2(s) + \text{const.}$$

Second B.V.P.:-

Here displacements are prescribed on the boundary, i.e., u_1 and u_2 are given on boundary.

Let $u_1 = g_1(s), \qquad u_2 = g_2(s)$

$\Rightarrow u_1 + iu_2 = g_1(s) + ig_2(s)$

But

$$2\mu(u_1 + iu_2) = \left[\kappa\phi(z) - z\overline{\phi}'(z) - \overline{\psi}'(z)\right]$$

$$\Rightarrow 2\mu(g_1 + ig_2) = \left[\kappa\phi(z) - z\overline{\phi}'(z) - \overline{\psi}'(z)\right]$$

is known on boundary.

Such types of problems are known as Second B.V.P.

2.8 Boundary conditions in terms of Normal and tangential components:

If Normal and tangential components of surface tractions are known on the boundary of the body, then prove that B.C.'s are expressed as:

$$N - iT = \phi'(z) + \overline{\phi}'(z) - e^{2i\alpha}\left[\bar{z}\phi''(z) + \psi''(z)\right] \qquad \text{on C (Boundary)}$$

where N is Normal component and

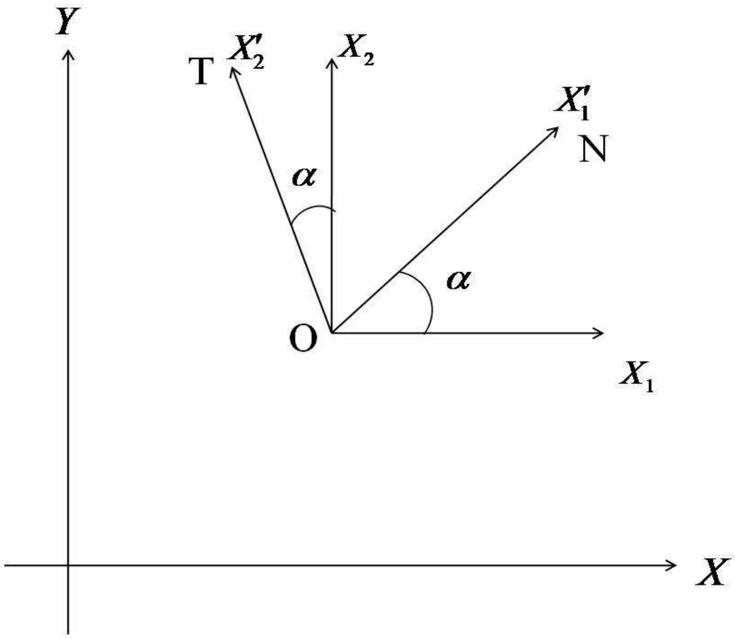
T is Tangential component,

α is angle measured from the positive direction of x_1-axis to the normal.

Proof: Here ox_1x_2 is one coordinate system.

$ox_1'x_2'$ is other system obtained by rotation from the coordinate system ox_1x_2.

($N = T_{11}', \; T = T_{12}'$ or $N = \tau_{11}', \; T = \tau_{12}'$)



From the tensorial character, we have

$$\tau'_{ij} = l_{ip} l_{jq} \tau_{pq} \quad ; \qquad i, j = 1, 2$$

Therefore

$$\tau'_{ii} = l_{ip} l_{iq} \tau_{pq} = \delta_{pq} \tau_{pq} = \tau_{pp} \ \ or \ \ \tau_{qq}$$

$$i.e., \ \tau'_{11} + \tau'_{22} = \tau_{11} + \tau_{22} \tag{1}$$

$$\begin{aligned} \tau'_{11} &= l_{1p} l_{1q} \tau_{pq} \\ &= l_{11} l_{1q} \tau_{1q} + l_{12} l_{1q} \tau_{2q} \\ &= l_{11} l_{11} \tau_{11} + l_{11} l_{12} \tau_{12} + l_{12} l_{11} \tau_{21} + l_{12} l_{12} \tau_{22} \\ &= l^2{}_{11} \tau_{11} + 2 l_{11} l_{12} \tau_{12} + l^2{}_{12} \tau_{22} \end{aligned}$$

Since $l_{ij} = \cos\left(x_i', x_j\right)$

Therefore

$$l_{11} = \cos\left(x_1', x_1\right) = \cos\alpha$$

$$l_{12} = \cos\left(x_1', x_2\right) = \sin\alpha$$

$$l_{21} = \cos\left(x_2', x_1\right) = -\sin\alpha$$

$$l_{22} = \cos\left(x_2', x_2\right) = \cos\alpha$$

$$\tau_{11}' = \cos^2\alpha\ \tau_{11} + 2\cos\alpha\sin\alpha\ \tau_{12} + \sin^2\alpha\ \tau_{22}$$

Similarly

$$\tau_{22}' = \sin^2\alpha\ \tau_{11} - 2\cos\alpha\sin\alpha\ \tau_{12} + \cos^2\alpha\ \tau_{22}$$

$$\tau_{12}' = l_{1p} l_{2q} \tau_{pq} = -\sin\alpha\cos\alpha\ \tau_{11} + \left(\cos^2\alpha - \sin^2\alpha\right)\ \tau_{12} + \sin\alpha\cos\alpha\ \tau_{22}$$

Now

$$\tau_{22}' - \tau_{11}' + 2i\tau_{12}' = -\left[\cos 2\alpha + i\sin 2\alpha\right]\tau_{11} - 2\left[\sin 2\alpha - i\cos 2\alpha\right]\tau_{12} + \left[\cos 2\alpha + i\sin 2\alpha\right]\tau_{22}$$

$$\begin{aligned}\Rightarrow \tau_{22}' - \tau_{11}' + 2i\tau_{12}' &= -e^{2i\alpha}\ \tau_{11} - \frac{2}{i}\left[\cos 2\alpha + i\sin 2\alpha\right]\tau_{12} + e^{2i\alpha}\ \tau_{22} \\ &= e^{2i\alpha}\left[\tau_{22} - \tau_{11} + 2i\tau_{12}\right]\end{aligned} \qquad (2)$$

Subtracting (2) from (1), we get

$$2\tau_{11}' - 2i\tau_{12}' = \left[\left(\tau_{11} + \tau_{22}\right) - e^{2i\alpha}\left(\tau_{22} - \tau_{11} + 2i\tau_{12}\right)\right]$$

but

$$\tau_{11} + \tau_{22} = 2\left[\phi'(z) + \bar{\phi}'(z)\right]$$

$$\tau_{22} - \tau_{11} + 2i\tau_{12} = 2\left[\bar{z}\phi''(z) + \psi''(z)\right]$$

Taking $\quad \tau_{11}' = N, \quad \tau_{12}' = T$

So, we have

$$2(N - iT) = 2\left[\phi'(z) + \bar{\phi}'(z) - e^{2i\alpha}\left(\bar{z}\phi''(z) + \psi''(z)\right)\right]$$

$$\Rightarrow N - iT = \phi'(z) + \bar{\phi}'(z) - e^{2i\alpha}\left(\bar{z}\phi''(z) + \psi''(z)\right)$$

which is the required result.

2.9 Summary

In this chapter we have discussed about the general solution of Biharmonic equation. We represented stresses and displacements in terms of complex potentials. We also discussed the arbitrariness in selection of functions $\phi(z)$ and $\psi(z)$ when the displacements or the stresses are given. We have derived first and second boundary value problems in plane elasticity.

2.10 Keywords Plane strain, Plane stress, Biharmonic function, analytic functions, Boundary value problems.

2.11 Self-assessment Questions

Q 1. Prove that the functional form of Airy's stress function Φ is :

$$2\Phi = \left[\bar{z}\,\psi_1(z) + z\,\overline{\psi_1}(z) + \psi_2(z) + \overline{\psi_2}(z)\right]$$

where $\psi_1(z)$ and $\psi_2(z)$ are two arbitrary analytic functions of complex variable.

Q 2. Starting with the equations of equilibrium, show that for plane strain conditions

$$2\mu(u_1 + iu_2) = \left[\kappa\phi(z) - z\bar{\phi}'(z) - \bar{\psi}'(z)\right],$$

where $\phi(z)$ and $\psi(z)$ are analytic functions and $\kappa = 3 - 4\sigma$, σ being the Poisson's ratio.

Q 3. Assuming plane strain conditions, obtain expressions for stresses τ_{11}, τ_{22} and τ_{12} in terms of two analytic functions.

Q 4. Prove that boundary conditions are expressed as:

$$N - iT = \phi'(z) + \bar{\phi}'(z) - e^{2i\alpha}\left[\bar{z}\phi''(z) + \psi''(z)\right] \qquad \text{on } C,$$

where Normal (N) and tangential (T) components of surface tractions are known on the boundary (C) of the body and α is angle measured from the positive direction of x_1-axis to the normal.

2.12 Suggested Readings

1. I.S. Sokolnikoff, Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company Ltd., New Delhi.
2. Y.C. Fung, Foundations of Solid Mechanics, Prentice Hall, New Delhi.
3. S. Timoshenko and N. Goodier, Theory of Elasticity, McGraw Hill, New York.
4. Martin H. Sadd., Elasticity Theory, Applications and Numerics AP (Elsevier).
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Chapter-3

Viscoelastic Models

3.1 Objectives

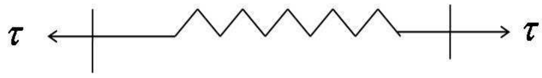
In this chapter, we shall discuss about some basic definitions related to Viscoelastic Materials and to derive constitutive equations for two viscoelastic models namely Maxwell and Kelvin. Further, the creep and relaxation phenomena will be discussed.

3.2 Introduction

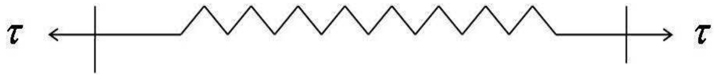
The property of the body to regain its original configuration (length, volume or shape) when the deforming forces are removed is called **elasticity**. The materials or substances which have property of elasticity are called elastic materials. For example, spring. For an elastic material there exists a one-to-one coordination between stress and strain. In the simplest case, there are six algebraic equations giving the strain components in terms of the stresses or vice versa. If they are linear, they are known as Hooke's law. Some materials show a pronounced influence of the rate of loading, the strain being larger if the stress has grown more slowly to its final value. The same materials display creep, that is, an increasing deformation under sustained load, the rate of strain depending on the stress. Such materials are called viscoelastic. The constitutive equations of these materials may be either linear or nonlinear. The viscoelastic materials are time dependent while elastic materials are time independent.

3.3 Viscoelastic Materials

Elastic material: The materials or substances which have property of elasticity are called elastic materials. For example spring, Elastic ball.



Spring before deformation



Spring after deformation

Then according to Hook's law, within elastic limit, the stress developed is directly proportional to the strain produced in a body, i.e.

stress α strain ,

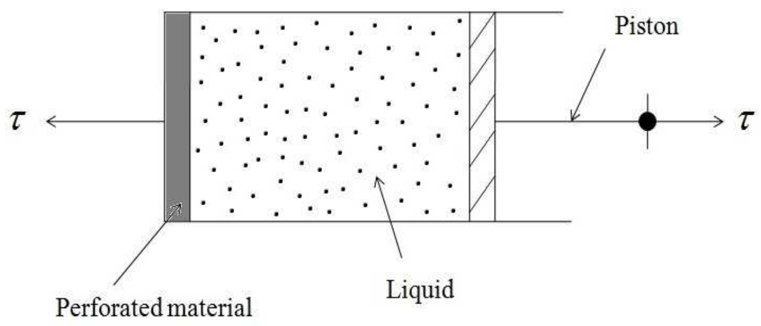
or stress = $E \times$ strain ,

i.e., $\tau = Ee$, where E is a constant and is known as Modulus of elasticity of the material of the body or Young's Modulus.

Viscosity: Viscosity is the property of a fluid (liquid or gas) by virtue of which an internal frictional force comes into play when the fluid in motion and opposes the relative motion of its different layers. It is also called fluid friction.

Viscous material: The materials having the property of viscosity are called viscous materials. For example: honey, dashpot.

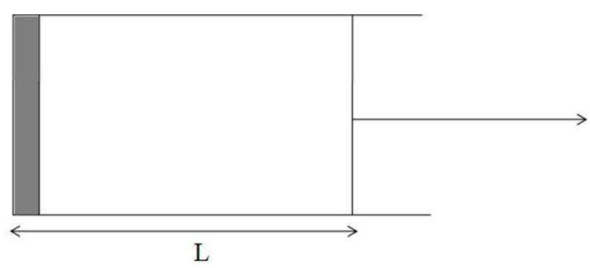
Dashpot: Consider the dashpot shown in figure below. A piston is moving in a cylinder with a perforated bottom so that no air is trapped inside. Between the cylinder and the piston wall, there is a rather viscous lubricant (liquid) so that a force is needed to displace the piston. The stronger this force, the faster the piston will move.

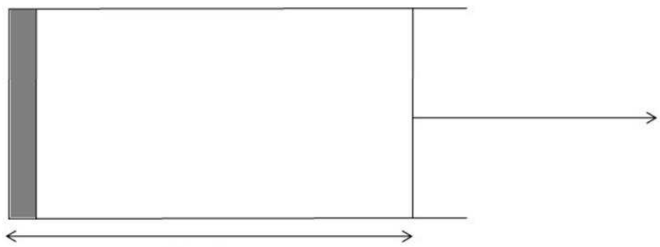


Dashpot

3.4 Governing equation for viscous material:

Let L be the original length of dashpot and l be the extension produced in the dashpot during deformation, then after deformation length of dashpot is L + l.





Now $\dfrac{\partial(l+L)}{\partial t}=\dfrac{\partial l}{\partial t}+\dfrac{\partial L}{\partial t}=\dfrac{\partial l}{\partial t}$ $\quad$ ($\because$ L is constant) $\quad$ (1)

Since $e=\dfrac{l}{L} \Rightarrow \quad l=eL$

$$\frac{\partial l}{\partial t}=\frac{\partial(eL)}{\partial t}=L\frac{\partial e}{\partial t}$$

Then (1) $\Rightarrow \dfrac{\partial(l+L)}{\partial t}=L\dfrac{\partial e}{\partial t}$

Let τ be the stress developed in dashpot, then we have

$$\tau \; \alpha \; \frac{\partial(l+L)}{\partial t}$$

$$\Rightarrow \tau=K\frac{\partial(l+L)}{\partial t}=KL\frac{\partial e}{\partial t}=\eta\frac{\partial e}{\partial t} \quad \text{where} \quad \eta=KL$$

Therefore in viscous medium, the basic governing equation is

stress α rate of strain, i.e.,

$$\tau \; \alpha \; \frac{\partial e}{\partial t} \quad \Rightarrow \quad \tau=\eta\frac{\partial e}{\partial t}=\eta\dot{e},$$

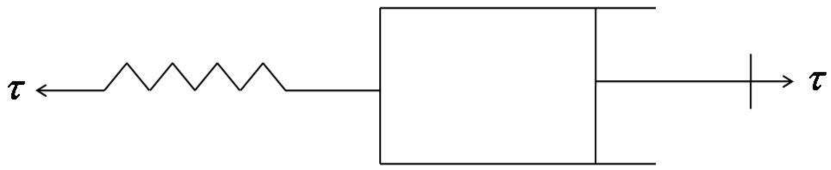
where η is coefficient of viscosity.

The quantity $\dot{e}$ is called the strain rate where dot represents ordinary or partial derivatives with respect to time t. Thus, a material whose stress is proportional to the strain rate is called a viscous material.

3.5 Three basic viscoelastic materials (or models)

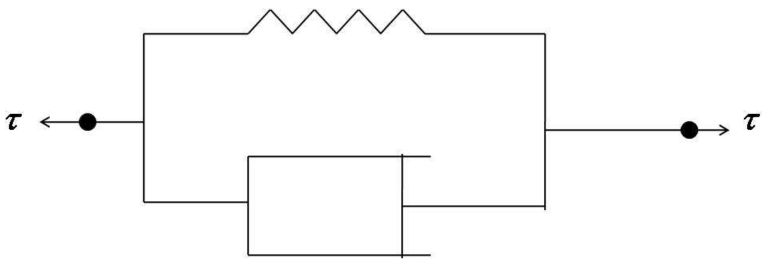
Linear viscoelastic materials are the combination of elastic and viscous materials. Viscoelastic materials (models) are constructed by the combining spring and dashpot.

a) **Maxwell Model (or Maxwell materials):-** In this model, spring and dashpot are connected in series. This model is also called Maxwell fluid.



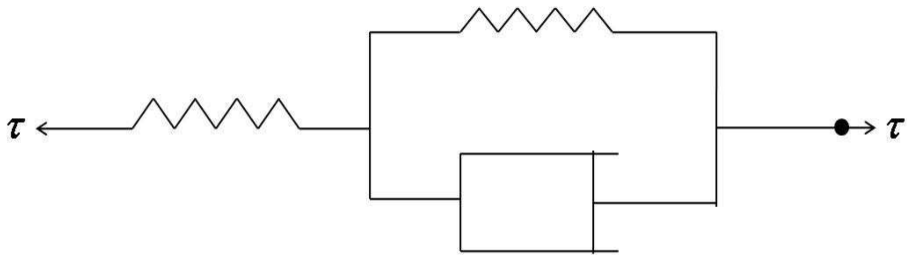
Maxwell model

b) **Kelvin Model: -** In this model, spring and dashpot are connected in parallel**.**



Kelvin model

c) **Standard Linear Solid (or three parameter solid):-** In this model, a spring is connected in series with a Kelvin's model.



Standard Linear Solid

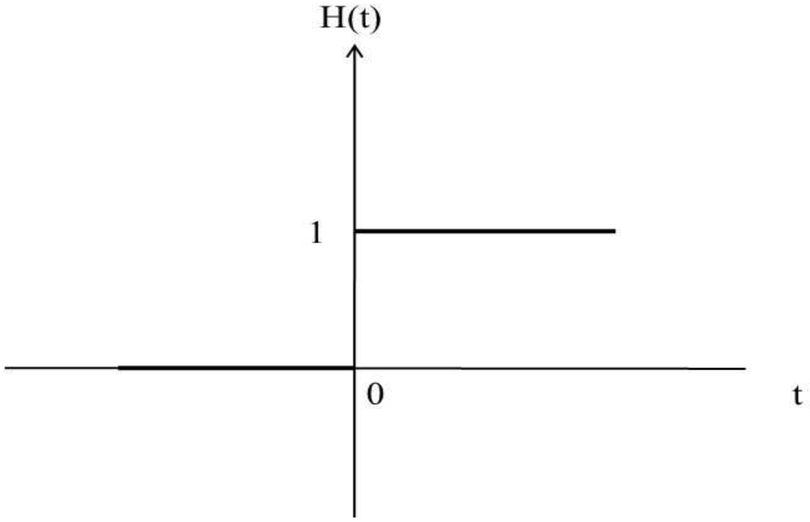
For every model, we shall consider following three things:

1. Constitutive equation (stress-strain relations)
2. Creep Phase
3. Relaxation Phase

Principle of Superposition: If stress τ_1 produces strain e_1 and stress τ_2 produces strain e_2, then the total stress $\tau_1 + \tau_2$ produces strain $e_1 + e_2$.

Heaviside's unit step function: It is denoted as $H(t)$ or $u(t)$ or $\Delta(t)$ and is defined as $H(t) = \begin{cases} 1 \ , & t > 0 \\ 0 \ , & t < 0 \end{cases}$

The function $H(t)$ is discontinuous at t = 0.



Dirac delta function: It is denoted as $\delta(t)$ and is defined as

$$\delta(t) = \begin{cases} 0 \ , & t \neq 0 \\ \infty \ , & t = 0 \end{cases}$$

Then $\int_{-\infty}^{\infty} \delta(t)\, dt \ = \int_{0^-}^{0^+} \delta(t)\, dt \ = \ 1$

Creep Phase: Creep is the slow increasing deformation of a material under constant stress and the rate of strain depends upon the stress.

For this, consider the stress cycle

$\tau(t) = \tau_0 H(t)$, where $H(t)$ is unit step function. So

$$\tau(t) \ = \begin{cases} \tau_0 \ , & t > 0 \\ 0 \ , & t < 0 \end{cases}$$

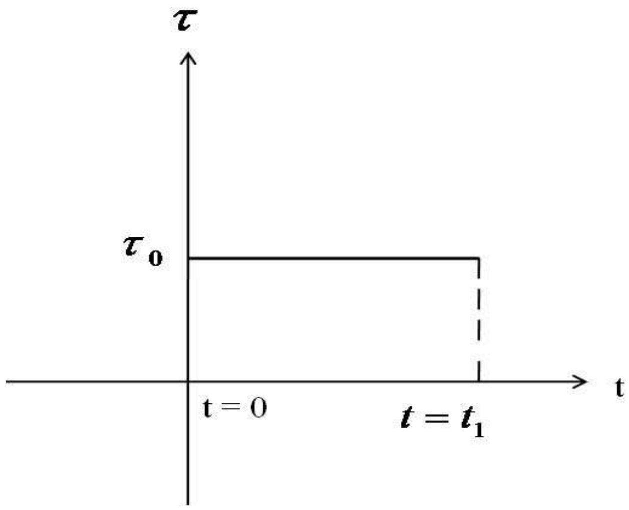
For an elastic material, the strain cycle is

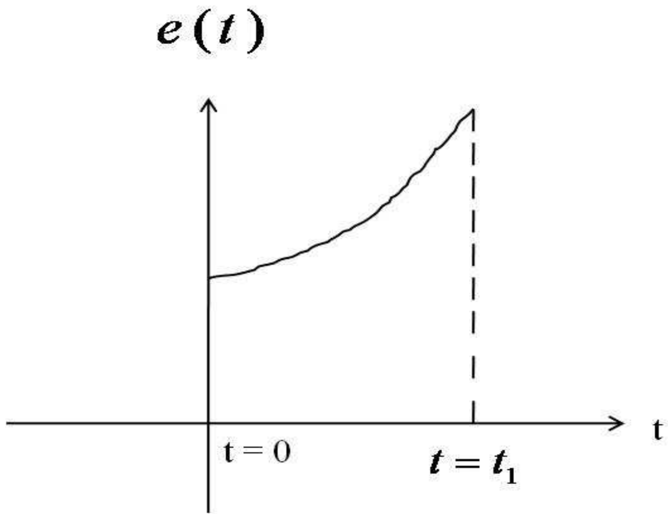
$e(t) = e_0 H(t)$, where $H(t)$ is unit step function. So

$$e(t) = \begin{cases} e_0 \ , & t > 0 \\ 0 \ , & t < 0 \end{cases}$$

But for viscoelastic material, the corresponding strain cycle is, $e(t) = \tau_0 J(t)$

where $J(t)$ an increasing function of t. $J(t)$ is different for different materials and is called creep compliance.





Relaxation Phase: Consider the strain cycle

$e(t) = e_0 H(t)$

$$= \begin{cases} e_0 \quad , \quad t > 0 \\ 0 \quad , \quad t < 0 \end{cases}$$

For an elastic material

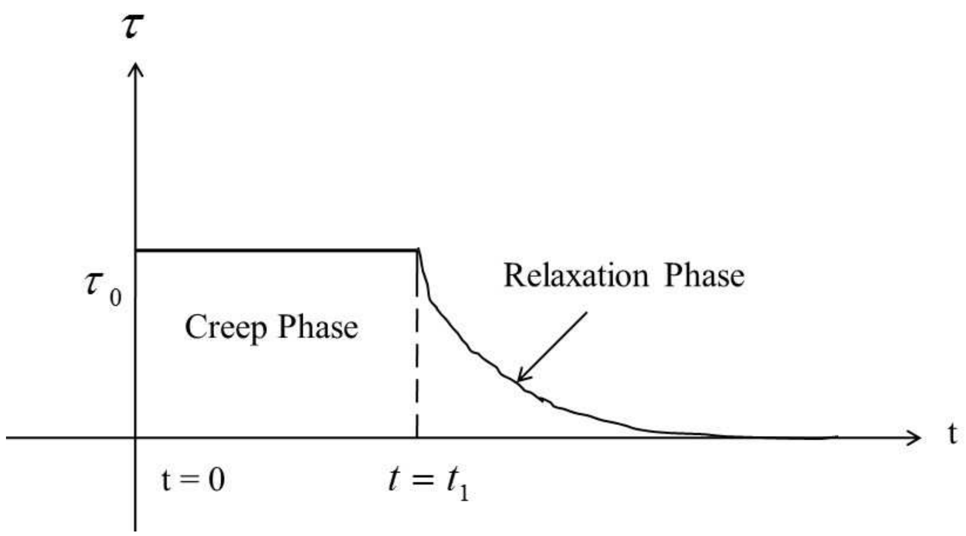
$\tau(t) = \tau_0 H(t)$

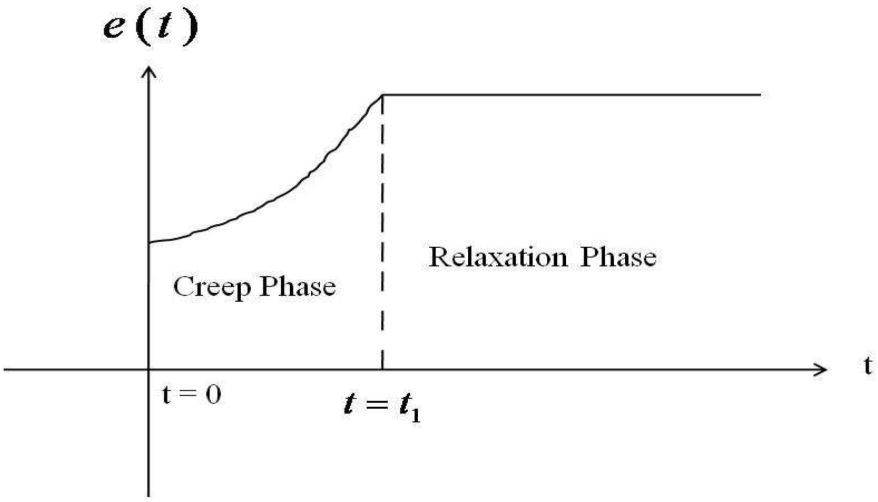
$$= \begin{cases} \tau_0 \quad , \quad t > 0 \\ 0 \quad , \quad t < 0 \end{cases}$$

For viscoelastic material, $\tau(t) = e_0 Y(t)$,where $Y(t) = 0,\ t < 0$

$Y(t)$ is called Relaxation Modulus.

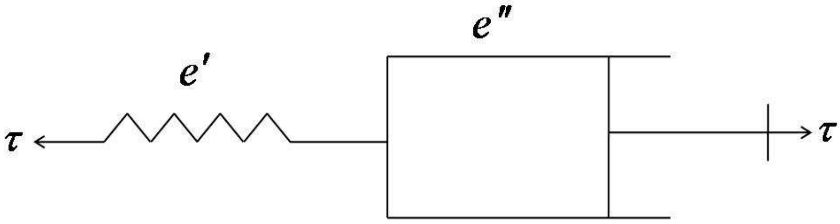
$Y(t)$ is a decreasing function of t and is different for different materials.





3.6 Maxwell Model

A spring and a dashpot are connected in series.



Since elements are connected in series. Hence, elongation is distributed on both elements.

If e is the total elongation then $e = e' + e''$ (1)

where e' is the elongation in the spring and e'' is the elongation in the dashpot.

The stress-strain relation for spring is $\tau = Ee'$ (2)

The stress-strain relation for dashpot is $\tau = \eta \dfrac{\partial e''}{\partial t}$ (3)

To obtain the stress-strain relation for the Maxwell Model, eliminating e', e'' from equations (1)-(3).

From (1), differentiate w.r.t. time t,

$\dot{e} = \dot{e}' + \dot{e}''$

or $\frac{\partial e}{\partial t} = \frac{1}{E}\frac{\partial \tau}{\partial t} + \frac{\tau}{\eta}$

or $\tau + \frac{\eta}{E}\dot{\tau} = \eta\dot{e}$ (4)

Comparing it with the standard stress-strain relation for a viscoelastic material

$$\sum_{k=0}^{m} p_k \frac{d^k \tau}{dt^k} = \sum_{k=0}^{m} q_k \frac{d^k e}{dt^k} \quad \text{with} \quad p_0 = 1$$

$$\tau + p_1\dot{\tau} + p_2\ddot{\tau} + ... = q_0 e + q_1\dot{e} + q_2\ddot{e} + ...$$

We have $p_1 = \frac{\eta}{E}$, $q_0 = 0$, $q_1 = \eta$

Equation (4) can be re-written as

$$\dot{\tau} + \frac{E}{\eta}\tau = E\dot{e} \quad (5)$$

Or $\dot{\tau} + \frac{\tau}{t^*} = E\dot{e}$, (6)

where $t^* = \frac{\eta}{E}$ is Relaxation time.

Equation (5) or (6) is required constitutive equation (or stress-strain relation) for a Maxwell model.

Creep Phase for Maxwell model:

Consider the stress cycle, i.e. we apply a constant stress at t = 0 and discuss the behaviour of strain.

$$\tau(t) = \tau_0 H(t)$$

$$= \begin{cases} \tau_0 & , \quad t > 0 \\ 0 & , \quad t < 0 \end{cases} \qquad (7)$$

From equation (5) and (7)

$$\frac{\partial \tau_0}{\partial t} + \frac{E}{\eta}\tau_0 = E\frac{\partial e}{\partial t}$$

$$\Rightarrow \frac{E}{\eta}\tau_0 = E\frac{\partial e}{\partial t}$$

$$\Rightarrow \frac{\partial e}{\partial t} = \frac{\tau_0}{\eta}$$

Integrating w.r.t. ‘t’ , we get

$$e(t) = \frac{\tau_0}{\eta}t + e_0 , \qquad (8)$$

where e_0 is constant of integration.

To find e_0, we integrate equation (6) w.r.t. time (t) between $(-\varepsilon, \varepsilon)$

$$\int_{-\varepsilon}^{\varepsilon} \frac{\partial \tau}{\partial t} dt + \int_{-\varepsilon}^{\varepsilon} \frac{\tau}{t^*} dt = E \int_{-\varepsilon}^{\varepsilon} \frac{\partial e}{\partial t} dt$$

$$\tau(\varepsilon) - \tau(-\varepsilon) + \left[\int_{-\varepsilon}^{0} 0 \; + \; \int_{0}^{\varepsilon} \frac{\tau_0}{t^*} \, dt \right] = E \, [e(\varepsilon) - e(-\varepsilon)]$$

But $\tau(-\varepsilon) = e(-\varepsilon) = 0$

Since material is in the natural state, therefore

$$\tau(\varepsilon)+\frac{\tau_0}{t^*}\varepsilon = E\ e(\varepsilon)$$

Taking limit as $\varepsilon \to 0^+$, therefore

$$\tau(0^+) = E\ e(0^+) \qquad (9)$$

Taking $t \to 0^+$ in equation (8), we get

$$e(0^+) = e_0$$

$$\Rightarrow \frac{\tau(0^+)}{E} = e_0 \qquad \text{(Using (9))}$$

$$\Rightarrow e_0 = \frac{\tau_0}{E}$$

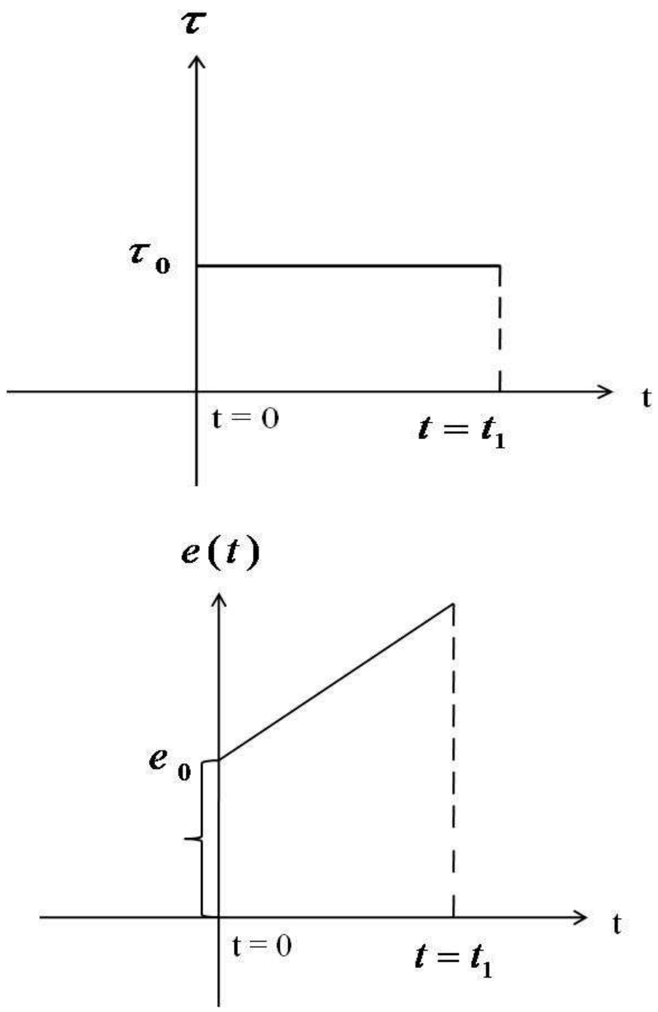
Using this value in equation (8), we get

$$e(t) = \frac{\tau_0}{\eta}t + \frac{\tau_0}{E} = \tau_0\left(\frac{t}{\eta}+\frac{1}{E}\right) = \frac{\tau_0}{\eta}\left(t+t^*\right)$$

Comparing with the definition of creep compliance

$$e(t) = \tau_0 J(t)$$

So, $J(t) = \frac{1}{\eta}\left(t+t^*\right)$ (10)



It is observed that for a fixed amount of stress, the strain instantly takes a finite value, which is the behaviour of an elastic solid. So, for large values of t, the deformation goes infinitely, which is behaviour of a viscous fluid.

Relaxation Phase for Maxwell model:

We assume that the strain cycle is given below and we discuss behaviour of stress under constant strain.

$e(t) = e_0 H(t)$

$$= \begin{cases} e_0 \ , & t > 0 \\ 0 \ , & t < 0 \end{cases} \qquad (11)$$

From equation (6), for t > 0, we get

$$\dot{\tau} + \frac{\tau}{t^*} = 0$$

I.F. = $e^{\int \frac{1}{t^*} dt} = e^{t/t^*}$

Solution is $\tau(t) e^{t/t^*} = \int 0\, dt +$ constant(τ_0)

$$\tau(t) = \tau_0 \, e^{-t/t^*}, \qquad (12)$$

where τ_0 is constant of integration.

Taking limit as $t \to 0^+$ in equation (12), we get

$$\tau(0^+) = \tau_0$$

$$\Rightarrow Ee(0^+) = \tau_0 \Rightarrow \tau_0 = Ee_0$$

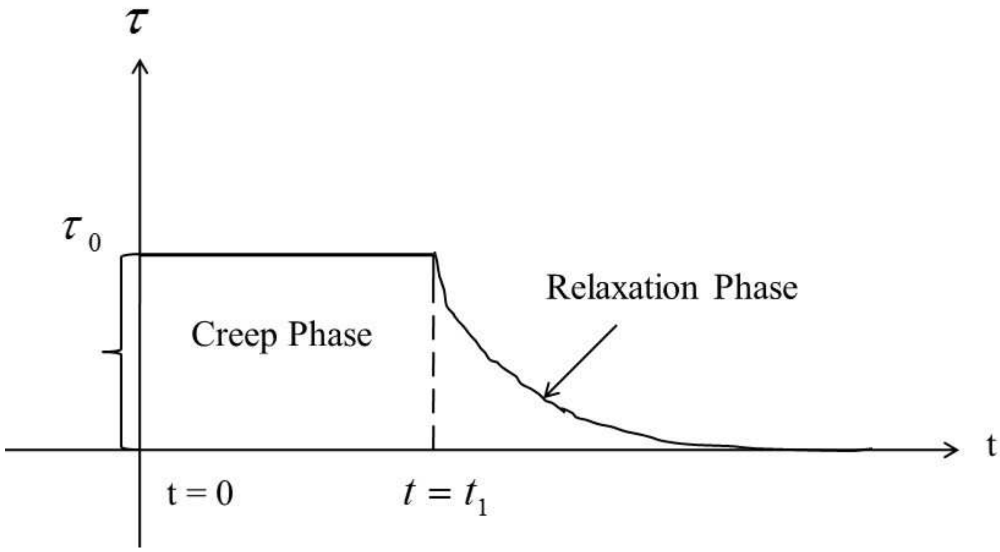
Substituting the value of τ_0 in equation (12), we get

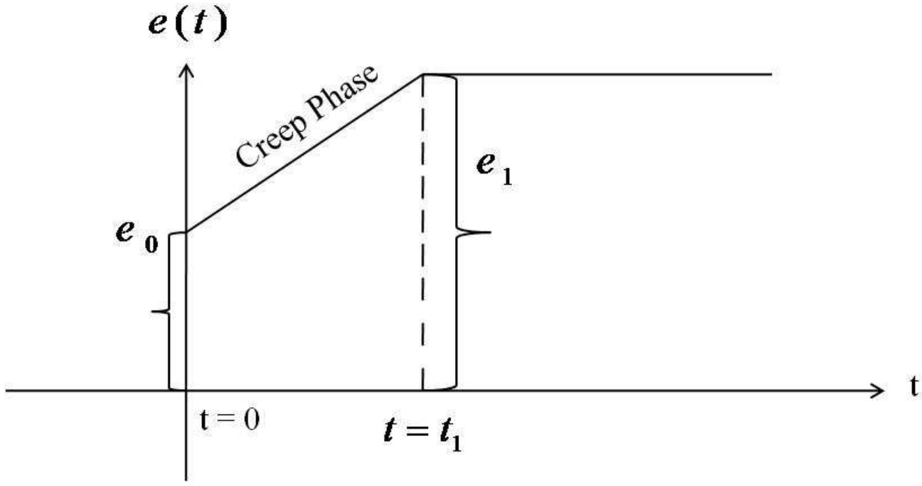
$$\tau(t) = E\, e_0 \, e^{-t/t^*} = e_0 \left[E e^{-t/t^*} \right] \qquad (13)$$

Equation (13) is required stress cycle.

Comparing with definition of $Y(t)$,

$$Y(t) = Ee^{-t/t^*}$$



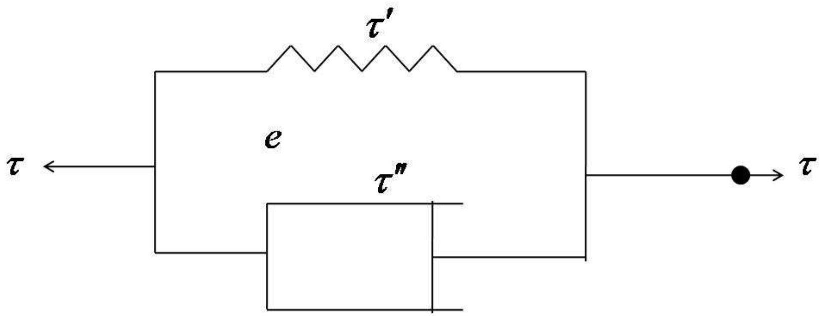


For a finite amount of strain, the Maxwell Model exhibits a finite amount of stress instantly and then it goes on decreasing.

For large values of t, the Maxwell material has complete Relaxation.

3.7 Kelvin Model

In this model a spring and a dashpot are connected in parallel.



Suppose model is acted upon by a force P causing stress τ. Sine elements are parallel, so τ is distributed itself upon both elements.

Hence $\tau = \tau' + \tau''$, (1)

where τ' is the stress on spring and τ'' is the stress on the dashpot.

Let e is the elongation of Kelvin element (model).

The stress-strain relation for a spring is $\tau' = Ee$ (2)

The stress-strain relation for a dashpot is $\tau'' = \eta \dfrac{\partial e}{\partial t}$ (3)

To obtain the stress-strain relation for the Kelvin element (model), eliminating τ' and τ'' from equations (1)-(3).

From (1)

$$\tau = Ee + \eta \dot{e} \tag{4}$$

Comparing it with the standard stress-strain relation for a viscoelastic material, we have

$q_0 = e$, $q_1 = \eta$

Equation (4) can be re-written as

$$\frac{\partial e}{\partial t}+\frac{E}{\eta}e=\frac{\tau}{\eta} \tag{5}$$

Or $\frac{\partial e}{\partial t}+\frac{e}{t^*}=\frac{\tau}{\eta}$ (6)

where $t^*=\frac{\eta}{E}$ = Relaxation time.

Equations (4) and (6) are required stress-strain relation for Kelvin Model.

Creep Phase:

Consider the stress cycle

$$\tau(t)=\tau_0 H(t)$$

$$=\begin{cases}\tau_0 & , \quad t>0\\ 0 & , \quad t<0\end{cases} \tag{7}$$

From equation (4) and (7), for t > 0

$$\tau_0 = Ee+\eta\frac{\partial e}{\partial t} \quad \text{or} \quad \frac{\partial e}{\partial t}+\frac{E}{\eta}e=\frac{\tau_0}{\eta}$$

Its integrating factor is given by

I.F. = $e^{\int \frac{1}{t^*}dt}=e^{t/t^*}$

So, solution of equation is

$$e(t)e^{t/t^*}=\int\frac{\tau_0}{\eta}e^{t/t^*}dt+c_1$$

$$= \frac{\tau_0}{\eta} t^* e^{t/t^*} + c_1$$

Or $e(t) = \frac{\tau_0}{E} + c_1 e^{-t/t^*}$ (8)

where $E = \frac{\eta}{t^*}$ and c_1 is constant of integration.

To find c_1, we integrate equation (5) w.r.t. time (t) between $(-\varepsilon, \varepsilon)$

$$\int_{-\varepsilon}^{\varepsilon} \frac{\partial e}{\partial t} dt + \frac{E}{\eta} \int_{-\varepsilon}^{\varepsilon} e(t) dt = \frac{1}{\eta} \int_{-\varepsilon}^{\varepsilon} \tau dt$$

$$e(\varepsilon) - e(-\varepsilon) + \frac{E}{\eta}\left[\int_{-\varepsilon}^{0} 0 + \int_{0}^{\varepsilon} e(t) dt\right] = \frac{1}{\eta}\left[\int_{-\varepsilon}^{0} 0 + \int_{0}^{\varepsilon} \tau_0 dt\right]$$

But $e(-\varepsilon) = 0$

Since material is in the natural state, therefore

$$e(\varepsilon) + \frac{E}{\eta} \int_{0}^{\varepsilon} e(t)\ dt = \frac{1}{\eta} \tau_0\ \varepsilon$$

Since viscoelastic material is a combination of elastic and viscous material. Hence, for

$\tau = \tau_0$ for t > 0, there is instant strain e_0 for t > 0.

So, $\int_{0}^{\varepsilon} e(t) dt = e_0 \varepsilon$

Therefore, $e(\varepsilon) + \frac{E}{\eta} e_0 \varepsilon = \frac{1}{\eta} \tau_0 \varepsilon$

Taking $\varepsilon \to 0$, we have $e(0^+) = 0$ (9)

Taking $t \to 0^+$ in equation (8), we get

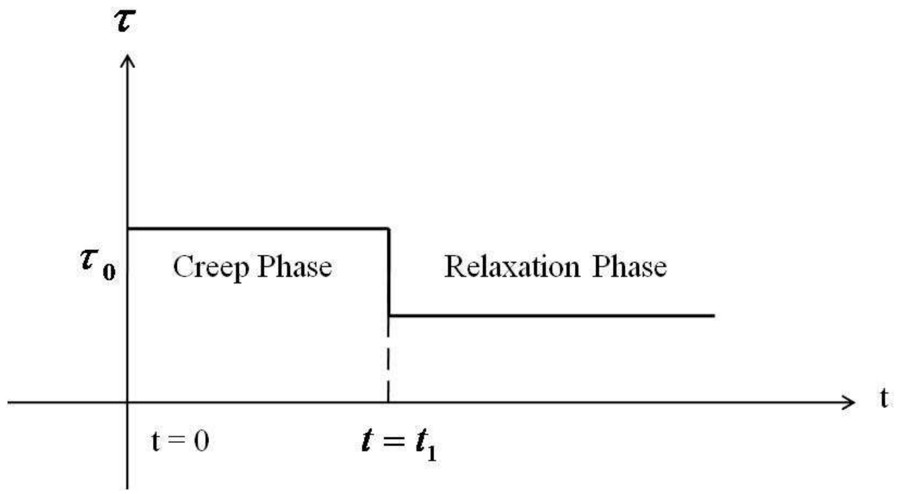
$$e(0^+) = \frac{\tau_0}{E} + c_1 = 0$$

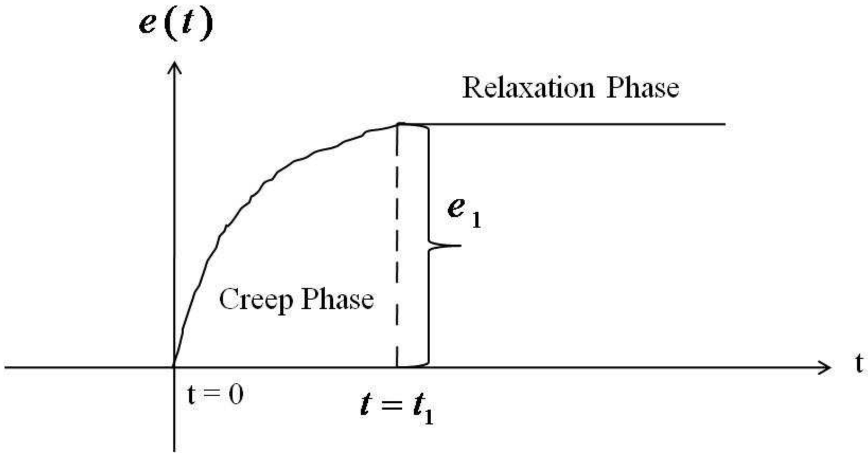
$$\Rightarrow c_1 = -\frac{\tau_0}{E} \qquad \text{(Using (9))}$$

Using this value in equation (8), we get

$$e(t) = \frac{\tau_0}{E} - \frac{\tau_0}{E} e^{-t/t^*} = \frac{\tau_0}{E}\left[1 - e^{-t/t^*}\right] \qquad (10)$$

This is the required strain cycle.





We observe that under finite load, the model initially deforms slowly. So, for small value of t, i.e., $t \to 0, e(t) = 0$.

For large value of t, i.e., $t \to \infty$,

$e(t) = \frac{\tau_0}{E} = e_\infty$, i.e., under finite stress, there is finite strain.

This is the behaviour of an elastic solid.

Sine element deform slowly, so Kelvin element has delayed elasticity.

Comparing with the definition of creep compliance

$$e(t) = \tau_0 J(t)$$

So, $J(t) = \frac{1}{E}\left(1 - e^{-t/t^*}\right)$

Relaxation Phase

Consider the strain cycle is

$$e(t) = e_0 H(t)$$

$$= \begin{cases} e_0 & , \quad t > 0 \\ 0 & , \quad t < 0 \end{cases}$$

It is not possible since Kelvin model does not attain finite strain instantaneously.

Suppose at $t = t_1 > 0$

$$e(t) = e_1 \text{ and } \quad e_1 = \frac{\tau_0}{E}\left[1 - e^{-t_1/t^*}\right] \qquad \text{(Using (10))} \qquad (11)$$

From equation (6), for t > 0 we get

$$0 + \frac{e_1}{t^*} = \frac{\tau}{\eta} \Rightarrow \tau(t) = \frac{\eta}{t^*} e_1 \Rightarrow \tau(t) = Ee_1$$

$$\Rightarrow \tau(t) = E\frac{\tau_0}{E}\left[1 - e^{-t_1/t^*}\right] = \tau_0\left[1 - e^{-t_1/t^*}\right] = \text{finite value}$$

We observe that $\tau(t)$ is independent of t.

The relaxation in Kelvin element is incomplete, since there is a stress forever.

Relaxation Modulus:

We have $\tau = Ee + \eta \dfrac{\partial e}{\partial t}$

If $e(t) = e_0 H(t)$

Then

$$\tau = Ee_0 H(t) + \eta \frac{\partial}{\partial t}(e_0 H(t)) = e_0[EH(t) + \eta\delta(t)]$$

$$\Rightarrow \frac{\tau}{e_0} = EH(t) + \eta\delta(t) = Y(t)$$

3.8 Summary

We have studied about elastic and viscoelastic materials. Constitutive equations of two viscoelastic models namely Maxwell and Kelvin have been derived. We have also discussed about creep and relaxation phenomena.

3.9 Keywords: Elastic material, Viscoelastic material, dashpot, Kelvin Model, Maxwell Model, creep and relaxation phase.

3.10 Self –assessment Questions

Q 1. Define elastic and viscoelastic materials along with example.

Q 2. Describe the Kelvin solid model of viscoelasticity. Find its constitutive equation and hence discuss the creep phase and relaxation phase.

Q 3. Describe the Maxwell solid model of viscoelasticity. Find its constitutive equation and hence discuss the creep phase and relaxation phase.

3.11 Suggested Readings

1. Y.C. Fung, Foundations of Solid Mechanics, Prentice Hall, New Delhi.
2. W. Flugge, Viscoelasticity, Springer Verlag.
3. R.M. Christensen, Theory of Viscoelasticity- An Introduction, 2nd Edition, 1982, Academic Press Inc., New York.
4. D.R. Bland, The Theory of Linear Viscoelasticity, Pergamon, New York, 1960.

Chapter 4

Standard Linear Solid and Generalised Viscoelastic Models

4.1 Objectives

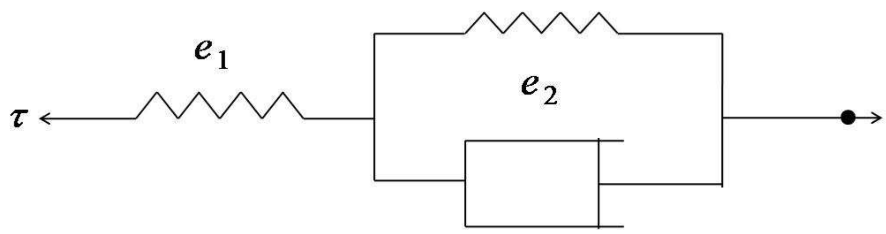
In this chapter, we shall discuss about more complicated models: Standard Linear Solid and Generalised Viscoelastic Models. We derive their constitutive equations. Further, the creep and relaxation phenomena will be explained.

4.2 Introduction

In Standard Linear Solid, a spring and a Kelvin model are connected in series. Further, there are two ways of systematically building up more complicated models: the Ke1vin chain and the Maxwell model. In the former, an arbitrary number of different Kelvin units are connected in series. In the Maxwell model, Maxwell units are connected in parallel. These are respectively called, the Generalized Maxwell Model and Generalized Kelvin Model.

4.3 Standard linear solid model (S.L.S.) or Three Parameter Solid

In SLS, a spring and a Kelvin model are connected in series.



Suppose model is under the action of applied force P causing stress τ. Since a spring and a Kelvin model are connected in series, so elongation is distributed itself over the spring and the Kelvin model.

If e is the total elongation, then $e = e_1 + e_2$ (1)

where e_1 is the elongation in the spring and e_2 is the elongation for the Kelvin model.

The stress-strain relation for spring is $\tau = E_1 e_1$ (2)

The stress-strain relation for the Kelvin model is $\tau = E_2 e_2 + \eta_2 \dot{e}_2$ (3)

Eliminating e_1 and e_2 from (1), (2) and (3).

$$\begin{aligned} E_2 e + \eta_2 \dot{e} &= E_2 (e_1 + e_2) + \eta_2 (\dot{e}_1 + \dot{e}_2) \\ &= E_2 e_1 + \eta_2 \dot{e}_1 + (E_2 e_2 + \eta_2 \dot{e}_2) \\ &= \frac{E_2 \tau}{E_1} + \eta_2 \frac{\dot{\tau}}{E_1} + \tau \\ &= \tau \left[1 + \frac{E_2}{E_1} \right] + \eta_2 \frac{\dot{\tau}}{E_1} \end{aligned}$$

$$\tau \left[\frac{E_1 + E_2}{E_1} \right] = E_2 e + \eta_2 \dot{e} - \eta_2 \frac{\dot{\tau}}{E_1}$$

$$\Rightarrow \tau + \frac{\eta_2 \dot{\tau}}{E_1 + E_2} = \frac{E_1 E_2}{E_1 + E_2} e + \frac{\eta_2 E_1}{E_1 + E_2} \dot{e}$$

$$\Rightarrow \tau + p_1 \dot{\tau} = q_0 e + q_1 \dot{e} \qquad (4)$$

where $p_1 = \dfrac{\eta_2}{E_1 + E_2}$, $q_0 = \dfrac{E_1 E_2}{E_1 + E_2}$, $q_1 = \dfrac{\eta_2 E_1}{E_1 + E_2}$ (5)

Equation (4) is the stress-strain relation for SLS.

Creep phase:

Consider the stress cycle

$\tau(t) = \tau_0 H(t)$, i.e.,

$$\tau(t) = \begin{cases} \tau_0 & , \quad t > 0 \\ 0 & , \quad t < 0 \end{cases} \qquad (6)$$

From equation (4) and (6), we have

$\tau_0 + 0 = q_0 e + q_1 \dot{e}$

$$\Rightarrow \frac{\partial e}{\partial t} + \frac{q_0}{q_1} e = \frac{\tau_0}{q_1} \qquad (7)$$

which is linear differential equation in e(t).

I.F. = $e^{\int \frac{q_0}{q_1} dt} = e^{\int \frac{1}{t_2^*} dt} = e^{t/t_2^*}$

The solution will be

$$e(t) e^{t/t_2^*} = \int \frac{\tau_0}{q_1} e^{t/t_2^*} dt + const.$$

$$e(t) e^{t/t_2^*} = \frac{\tau_0}{q_1} e^{t/t_2^*} t_2^* + const.$$

$$e(t) = \frac{\tau_0}{q_1} t_2^* + c_1 e^{-t/t_2^*}$$

$$e(t) = \frac{\tau_0}{q_1} \frac{q_1}{q_0} + c_1 e^{-t/t_2^*} = \frac{\tau_0}{q_0} + c_1 e^{-t/t_2^*} \qquad (8)$$

where c_1 is the constant of integration.

To find c_1:

We integrate equation (4) w.r.t. 't' between $(-\varepsilon, \varepsilon)$,

$$\int_{-\varepsilon}^{\varepsilon} \tau(t)dt + p_1 \int_{-\varepsilon}^{\varepsilon} \dot{\tau}(t)dt = q_0 \int_{-\varepsilon}^{\varepsilon} e(t)dt + q_1 \int_{-\varepsilon}^{\varepsilon} \dot{e}(t)dt$$

$$\Rightarrow \int_{-\varepsilon}^{0} 0dt + \int_{0}^{\varepsilon} \tau_0 dt + p_1 \tau(t)\Big|_{-\varepsilon}^{\varepsilon} = q_0 \int_{-\varepsilon}^{0} 0dt + q_0 \int_{0}^{\varepsilon} e(t)dt + q_1 e(t)\Big|_{-\varepsilon}^{\varepsilon}$$

$$\Rightarrow \tau_0 \varepsilon + p_1(\tau(\varepsilon) - \tau(-\varepsilon)) = q_0 \int_{0}^{\varepsilon} e(t)dt + q_1[e(\varepsilon) - e(-\varepsilon)]$$

$$\Rightarrow \tau_0 \varepsilon + p_1 \tau(\varepsilon) = q_0 \int_{0}^{\varepsilon} e(t)dt + q_1 e(\varepsilon)$$

$$\Rightarrow \tau_0 \varepsilon + p_1 \tau(\varepsilon) = q_0 \varepsilon(\textit{finite quantity}) + q_1 e(\varepsilon)$$

Taking $\varepsilon \to 0$, we have

$$p_1 \tau(0^+) = q_0 \times 0 + q_1 e(0^+)$$

$$\Rightarrow p_1 \tau(0^+) = q_1 e(0^+) \qquad (9)$$

and $\tau(0^+) = \tau_0$

From equation (8), taking $t \to 0^+$ and using equation (9),

We get $e(0^+) = \dfrac{\tau_0}{q_0} + c_1 = \dfrac{p_1 \tau_0}{q_1}$

$$\Rightarrow c_1 = \frac{p_1 \tau_0}{q_1} - \frac{\tau_0}{q_0} = -\frac{\tau_0}{q_0}\left(1 - \frac{p_1 q_0}{q_1}\right)$$

As we have $\dfrac{p_1 q_0}{q_1} = \dfrac{\eta_2}{E_1 + E_2} \times \dfrac{E_2}{\eta_2} = \dfrac{E_2}{E_1 + E_2} < 1$

Putting value of c_1 in equation (8), we get

$$e(t) = \frac{\tau_0}{q_0} - \frac{\tau_0}{q_0}\left(1 - \frac{p_1 q_0}{q_1}\right) e^{-t/t_2^*}$$

$$\Rightarrow e(t) = \frac{\tau_0}{q_0}\left[1 - \left(1 - \frac{p_1 q_0}{q_1}\right) e^{-t/t_2^*}\right] \tag{10}$$

which is required strain cycle.

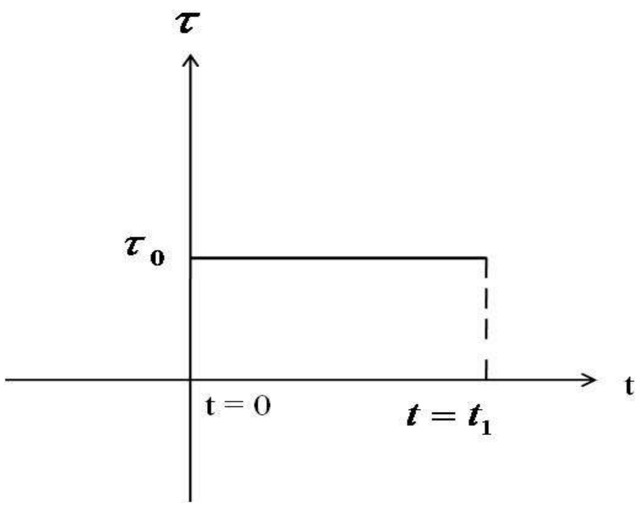
$$e(0) = \frac{\tau_0}{q_0} \times \frac{p_1 q_0}{q_1} = \frac{\tau_0 p_1}{q_1}$$

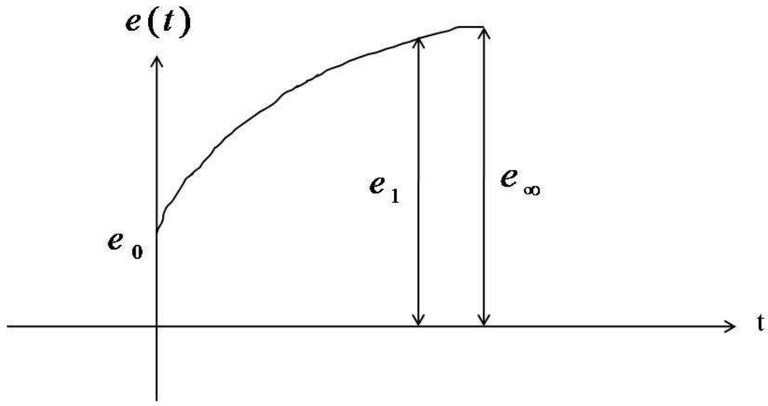
$$e(\infty) = \frac{\tau_0}{q_0}$$

Since $\dfrac{p_1 q_0}{q_1} < 1 \quad \Rightarrow \quad \dfrac{p_1}{q_1} < \dfrac{1}{q_0}$

$$\Rightarrow \frac{p_1 \tau_0}{q_1} < \frac{\tau_0}{q_0} \quad \Rightarrow \quad e(0) < e(\infty)$$

Under finite stress for small values of t, the material attains a finite strain and then it increases. For large values of t there is a finite deformation which is the behaviour of an elastic solid. Hence, SLS is also called as three parameter solid.





Creep compliance is given by

$$\frac{e(t)}{\tau_0} = J(t) = \frac{1}{q_0}\left[1-\left(1-\frac{p_1 q_0}{q_1}\right)e^{-t/t_2^*}\right]$$

Relaxation Phase

Consider the strain cycle is

$e(t) = e_0 H(t)$, i.e.,

$$e(t) = \begin{cases} e_0 \ , & t > 0 \\ 0 \ , & t < 0 \end{cases} \tag{11}$$

From equation (4) and (11), we have

$$\tau + p_1\dot{\tau} = q_0 e_0 + q_1 0$$

$$\Rightarrow \frac{\partial \tau}{\partial t} + \frac{1}{p_1}\tau = \frac{q_0 e_0}{p_1}$$ which is Linear differential equation.

$$\text{I.F.} = e^{\int \frac{1}{p_1} dt} = e^{t/p_1}$$

$$\tau(t) e^{t/p_1} = \int \frac{q_0}{p_1} e_0 e^{t/p_1} dt + const.$$

$$\Rightarrow \tau(t) = q_0 e_0 + c_2 e^{-t/p_1} \qquad (12)$$

where c_2 is the constant of integration.

Taking $t \to 0^+$ in equation (12) and using equation (9), we get

$$\tau(0^+) = q_0 e_0 + c_2 = \frac{q_1 e(0^+)}{p_1} = \frac{q_1 e_0}{p_1}$$

$$\Rightarrow c_2 = \frac{q_1 e_0}{p_1} - q_0 e_0 = -q_0 e_0 \left(1 - \frac{q_1}{p_1 q_0}\right)$$

Putting value of c_2 in equation (12), we have

$$\tau(t) = q_0 e_0 \left[1 - \left(1 - \frac{q_1}{p_1 q_0}\right) e^{-t/p_1}\right]$$

which is required stress-cycle.

Relaxation Modulus is given by

$$Y(t) = \frac{\tau(t)}{e_0} = q_0\left[1 - \left(1 - \frac{q_1}{p_1 q_0}\right)e^{-t/p_1}\right]$$

Now $\tau(0) = q_0 e_0 \times \frac{q_1}{p_1 q_0} = e_0 \frac{q_1}{p_1}$

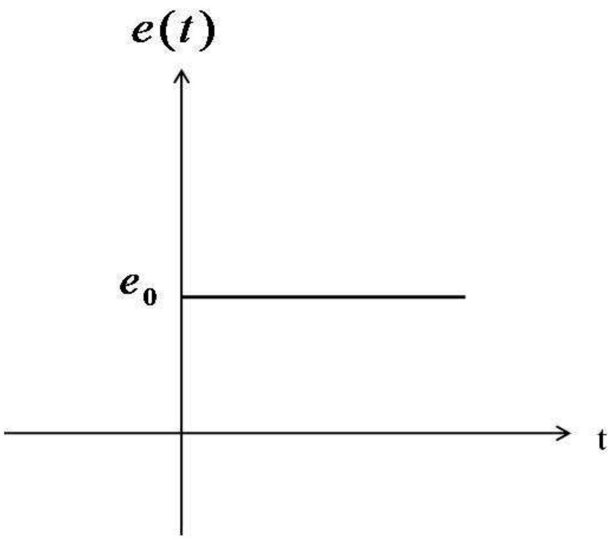
$\tau(\infty) = q_0 e_0$

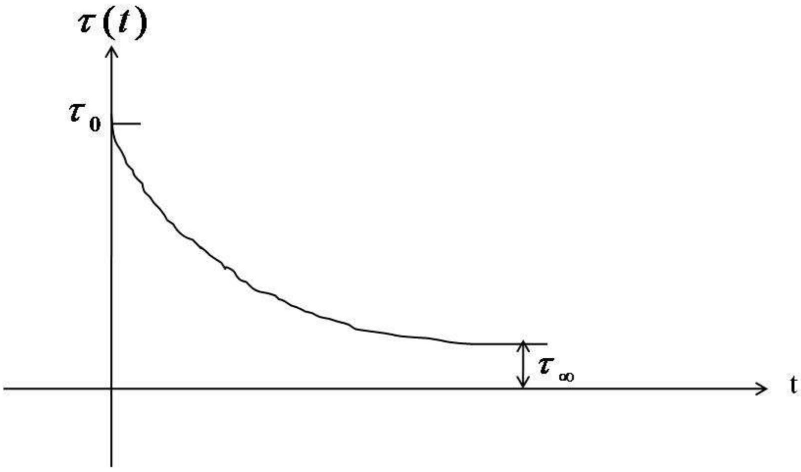
Since

$$\frac{p_1 q_0}{q_1} < 1 \quad \Rightarrow \quad \frac{p_1}{q_1} < \frac{1}{q_0}$$

$$\Rightarrow q_0 < \frac{q_1}{p_1} \quad \Rightarrow \quad q_0 e_0 < \frac{q_1 e_0}{p_1}$$

$\Rightarrow \tau(\infty) < \tau(0)$





Remark:

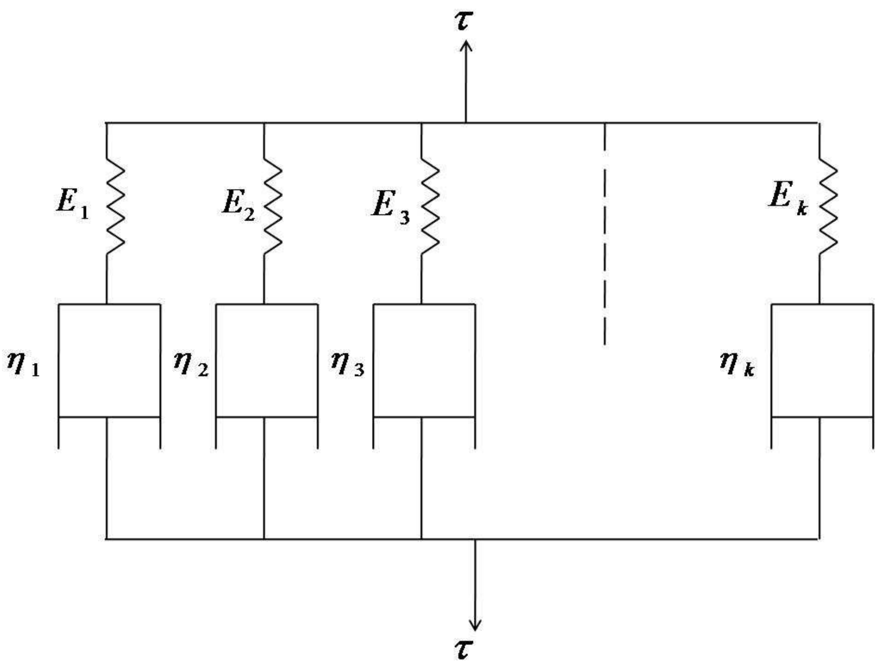
If we take , $p_1 = t_1^*$, $\frac{q_1}{q_0} = t_2^*$

$$\Rightarrow J(t) = \frac{1}{q_0}\left[1 - \left(1 - \frac{t_1^*}{t_2^*}\right)e^{-t/t_2^*}\right]$$

and $Y(t) = q_0\left[1 + \left(\frac{t_2^*}{t_1^*} - 1\right)e^{-t/t_1^*}\right]$

4.4 Generalized Maxwell Model

In Generalized Maxwell Model, *k* Maxwell elements are connected in parallel.



Let e_i be elongation in each Maxwell element due to applied stress τ. Since elongation will be same in each element. So,

$$e_1 = e_2 = ...= e_k = e\,(\text{say}) \qquad (1)$$

The stress τ distributes itself over the k elements.

Therefore, $$\tau = \sum_{r=1}^{k} \tau_r \qquad (2)$$

The Stress-Strain relation for rth Maxwell Model

$$\frac{\partial \tau_r}{\partial t} + \frac{\tau_r}{t^*{}_r} = E_r \frac{\partial e_r}{\partial t} = E_r \frac{\partial e}{\partial t} \qquad (3)$$

To obtain Stress-Strain relation for Generalized Maxwell Model, we eliminate $\tau_1, \tau_2, ...\tau_k$ from equations (2) and (3). We use the method of Laplace transform.

We denote $L\left(f(t)\right) = \bar{f}(s)$

Taking the Laplace transform of equations (2) and (3)

$$\bar{\tau}(s) = \bar{\tau}_1(s) + \bar{\tau}_2(s) + ... + \bar{\tau}_k(s).$$

$$\bar{\tau}(s) = \sum_{r=1}^{k} \bar{\tau}_r(s) \qquad (4)$$

$$L\left(\frac{\partial e}{\partial t}\right) = s\bar{e}(s) - e(0)$$

$$s\,\bar{\tau}_r(s) - \tau_r(0) + \frac{\bar{\tau}_r(s)}{t_r^*} = E_r\left[s\bar{e}(s) - e(0)\right]$$

Since material in the natural state at t = 0

$$\tau_r(0) = e(0) = 0$$

$$\bar{\tau}_r(s)\left[s + \frac{1}{t_r^*}\right] = E_r\left[s\bar{e}(s)\right]$$

$$\bar{\tau}_r(s) = E_r\left[\frac{s\bar{e}(s)}{s + \dfrac{1}{t_r^*}}\right]$$

Using in equation (4), we have

$$\bar{\tau}(s) = \sum_{r=1}^{k} E_r\left[\frac{s\bar{e}(s)}{s + \dfrac{1}{t_r^*}}\right]$$

$$\bar{\tau}(s) = s\bar{e}(s)\sum_{r=1}^{k}\left[\frac{E_r}{s+\frac{1}{t_r^*}}\right]$$

To find $\tau(t)$, we take the inverse Laplace transform and by using Convolution theorem

$$\tau(t) = L^{-}\left[s\bar{e}(s)\sum_{r=1}^{k}\frac{E_r}{s+\frac{1}{t_r^*}}\right] = \int_0^t \frac{\partial e(t')}{\partial t'}\left[\sum_{r=1}^{k} E_r e^{-(t-t')/t_r^*}\right]dt' \qquad (5)$$

Creep phase:

This is same as we calculated for single element.

Relaxation phase:

Consider the strain cycle is

$e(t) = e_0 H(t)$, i.e.,

$$e(t) = \begin{cases} e_0 \ , & t > 0 \\ 0 \ , & t < 0 \end{cases} \qquad (6)$$

From equation (3), $t > 0$

$$\frac{\partial \tau_r}{\partial t} + \frac{1}{t_r^*}\tau_r = E_r\frac{\partial e_0}{\partial t} = 0$$

Integrating above equation

Solution is $\tau_r(t)e^{t/t_r^*} = c_r$

$$\tau_r(t) = c_r e^{-t/t_r^*} \tag{7}$$

where c_r is constant of integration.

Taking $t = 0$ in equation (7), we get

$$\tau_r(0) = c_r \tag{8}$$

We integrate equation (3) w.r.t. time (t) between $(-\varepsilon, \varepsilon)$

$$\int_{-\varepsilon}^{\varepsilon} \frac{\partial \tau_r}{\partial t} dt + \int_{-\varepsilon}^{\varepsilon} \frac{\tau_r}{t_r^*} dt = E_r \int_{-\varepsilon}^{\varepsilon} \frac{\partial e_r}{\partial t} dt$$

$$\tau_r(\varepsilon) - \tau_r(-\varepsilon) + \left[\int_{-\epsilon}^{0} 0 + \int_{0}^{\epsilon} \frac{\tau_r}{t_r^*} dt \right] = E_r [e(\varepsilon) - e(-\varepsilon)]$$

But $\tau_r(-\varepsilon) = e(-\varepsilon) = 0$

Taking $\varepsilon \to 0$, we have

$$\tau_r(0^+) = E_r e(0^+) \tag{9}$$

Taking $t \to 0^+$ in equation (6), we get

$$e(0^+) = e_0$$

$$\Rightarrow \frac{\tau_r(0^+)}{E_r} = e_0 \qquad \text{(Using (9))}$$

$$\Rightarrow c_r = e_0 E_r \qquad \text{(Using equation (8))}$$

Hence equation (7) gives

$$\tau_r(t) = e_0 E_r e^{-t/t_r^*}$$

The stress cycle for Generalized Maxwell Model

$$\tau(t) = \sum_{r=1}^{k} \tau_r(t) = e_0 \sum_{r=1}^{k} E_r e^{-t/t_r^*}$$

Comparing with the definition of Relaxation Modulus

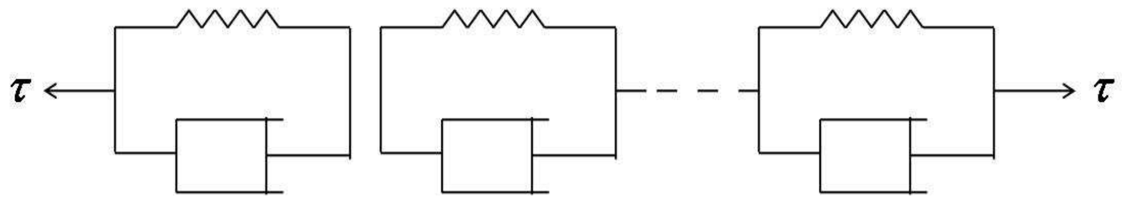
$$Y(t) = \frac{\tau(t)}{e_0} = \sum_{r=1}^{k} E_r e^{-t/t_r^*}$$

The stress-strain relation can be written as

$$\tau(t) = \int_0^t \frac{\partial e(t')}{\partial t'} Y(t - t')dt'$$

4.5 Generalized Kelvin Model

In Generalized Kelvin Model, '*k*' Kelvin elements are connected in series under the applied stress τ .



Since the elements are connected in series. So, elongation e is distributed on each element.

If $e_1, e_2, ..., e_k$ are the elongation of 1st, 2nd and *k*th Kelvin element, then

$$e = \sum_{r=1}^{k} e_r \tag{1}$$

The stress τ will be same over each element.

The Stress-Strain relation for rth Kelvin Model is

$$\tau = \eta_r \left(\dot{e}_r + \frac{e_r}{t_r^*} \right) \tag{2}$$

To obtain Stress-Strain relation for Generalized Kelvin Model, we eliminate

$e_1, e_2, ...e_k$ from equations (1) and (2). We use the method of Laplace transform.

We denote $L(f(t)) = \bar{f}(s)$

Taking the Laplace transform of equations (1) and (2)

$$\bar{e}(s) = \sum_{r=1}^{k} \bar{e}_r(s) \tag{3}$$

$$\bar{\tau}(s) = \eta_r \left[s\bar{e}_r(s) - e_r(0) + \frac{\bar{e}_r(s)}{t_r^*} \right]$$

But $e_r(0) = 0$, since material in the natural state at t = 0

$$\bar{\tau}(s) = \eta_r \bar{e}_r(s) \left[s + \frac{1}{t_r^*} \right]$$

Or $$\bar{e}_r(s) = \frac{\bar{\tau}(s)}{\eta_r \left[s + \frac{1}{t_r^*} \right]} \tag{4}$$

Using equation (4) in equation (3), we have

$$\bar{e}(s) = \bar{\tau}(s)\sum_{r=1}^{k}\frac{1}{\eta_r\left[s+\frac{1}{t_r^*}\right]}$$

To find $e(t)$, we take the inverse Laplace transform and by using Convolution theorem

$$e(t) = \int_0^t \tau(t')\left[\sum_{r=1}^{k}\frac{1}{\eta_r}e^{-(t-t')/t_r^*}\right]dt' \tag{5}$$

This is the required stress-strain relation for the Generalized Kelvin model.

Creep phase

Consider the stress cycle

$\tau(t) = \tau_0 H(t)$, i.e.,

$$\tau(t) = \begin{cases} \tau_0 & , \quad t > 0 \\ 0 & , \quad t < 0 \end{cases} \tag{6}$$

From equation (2), t > 0

$$\dot{e}_r + \frac{e_r}{t_r^*} = \frac{\tau_0}{\eta_r}$$

To integrate it, we have

I.F. = $e^{\int \frac{1}{t_r^*}dt} = e^{t/t_r^*}$

The solution will be

$$e_r(t)e^{t/t_r^*} = \int \frac{\tau_0}{\eta_r}e^{t/t_r^*}dt + const.$$

$$e_r(t)e^{t/t_r^*} = \frac{\tau_0}{\eta_r}\frac{e^{t/t_r^*}}{1/t_r^*} + const.$$

$$e_r(t) = \frac{\tau_0 t_r^*}{\eta_r} + c_r e^{-t/t_r^*} \qquad (7)$$

where c_r is the constant of integration.

To find c_r:

We take $t \to 0$

So, $e_r(0^+) = \frac{\tau_0 \eta_r}{\eta_r E_r} + c_r = \frac{\tau_0}{E_r} + c_r \qquad (8)$

We integrate equation (2) w.r.t. time (t) between $(-\varepsilon, \varepsilon)$

$$e_r(\varepsilon) - e_r(-\varepsilon) + (\,finite\,quantity)_\varepsilon = \frac{\tau_0}{\eta_r}\varepsilon$$

Making $\varepsilon \to 0$ and $e_r(-\varepsilon) = 0 \Rightarrow e_r(0^+) = 0$

Using in equation (8), we have

$$\frac{\tau_0}{E_r} + c_r = 0 \Rightarrow c_r = -\frac{\tau_0}{E_r}$$

Put the value in equation (7), we obtain

$$e_r(t) = \frac{\tau_0}{E_r} - \frac{\tau_0}{E_r}e^{-t/t_r^*} = \frac{\tau_0}{E_r}\left(1 - e^{-t/t_r^*}\right)$$

Then $e(t) = \tau_0 \sum_{r=1}^{k} \left(\frac{1 - e^{-t/t_r^*}}{E_r} \right)$

which is required strain cycle due to the stress-cycle $\tau(t) = \tau_0 H(t)$

Creep compliance is obtained as under:

$$J(t) = \frac{e(t)}{\tau_0} = \sum_{r=1}^{k} \left(\frac{1 - e^{-t/t_r^*}}{E_r} \right)$$

Differentiate w.r.t. 't',

$$\frac{dJ(t)}{dt} = \sum_{r=1}^{k} \left(\frac{-e^{-t/t_r^*}}{E_r} \times \frac{-1}{t_r^*} \right) = \sum_{r=1}^{k} \left(\frac{e^{-t/t_r^*}}{\eta_r} \right)$$

The stress-strain relation can be written as

$$e(t) = \int_0^t \tau(t') \frac{d}{d(t-t')} J(t-t') dt' \tag{9}$$

Example:-

A viscoelastic material is represented by a chain of 3 Kelvin elements.

Let q be reference of stress and T be reference time.

Assume that the following viscoelastic co-officiants hold for the Kelvin elements.

Ist element $q_0 = 2q$, $q_1 = 2qT$

2nd element $q_0 = q$, $q_1 = 4qT$

3rd element $q_0 = 1.5q$, $q_1 = 16.5qT$

Find $J(t), \bar{J}(s)$

The standard stress-strain relation for a Kelvin element is

$$\tau = q_0 e + q_1 \dot{e} = Ee + \eta\dot{e} \tag{1}$$

For Ist element $E_1 = 2q$, $\eta_1 = 2qT$

2nd element $E_2 = q$, $\eta_2 = 4qT$

3rd element $E_3 = 1.5q$, $\eta_3 = 16.5qT$

We have $\tau(t) = \tau_0 H(t)$

$$\bar{\tau}(s) = \frac{\tau_0}{s} \tag{2}$$

By definition of creep compliance,

$$e(t) = \tau_0 J(t)$$

$$\bar{e}(s) = \tau_0 \bar{J}(s) \tag{3}$$

For the Kelvin chain, we have

$$\bar{e}(s) = \bar{\tau}(s) \sum_{r=1}^{k} \frac{1}{\eta_r \left[s + \frac{1}{t_r^*} \right]} \tag{4}$$

$$t_1^* = \frac{\eta_1}{E_1} = T$$

$$t_2^* = \frac{\eta_2}{E_2} = 4T$$

$$t_3^* = \frac{\eta_3}{E_3} = 11T$$

Using equation (2) and (3) and the values of η_r, t_r^* in equation (4), we have

$$\tau_0 \bar{J}(s) = \frac{\tau_0}{s}\left[\frac{1}{2qT\left(s+\frac{1}{T}\right)} + \frac{1}{4qT\left(s+\frac{1}{4T}\right)} + \frac{1}{16.5qT\left(s+\frac{1}{11T}\right)}\right]$$

$$\bar{J}(s) = \frac{1}{qTs}\left[\frac{1}{2\left(s+\frac{1}{T}\right)} + \frac{1}{4\left(s+\frac{1}{4T}\right)} + \frac{1}{16.5\left(s+\frac{1}{11T}\right)}\right]$$

(To find J (t), we will take the inverse Laplace Transform)

$$\bar{J}(s) = \frac{1}{q}\left[\frac{1}{2T}\left(\frac{T}{s} - \frac{T}{s+\frac{1}{T}}\right) + \frac{1}{4T}\left(\frac{4T}{s} - \frac{4T}{s+\frac{1}{4T}}\right) + \frac{1}{16.5T}\left(\frac{11T}{s} - \frac{11T}{s+\frac{1}{11T}}\right)\right]$$

$$\bar{J}(s) = \frac{1}{q}\left[\frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+\frac{1}{T}}\right) + \left(\frac{1}{s} - \frac{1}{s+\frac{1}{4T}}\right) + \frac{2}{3}\left(\frac{1}{s} - \frac{1}{s+\frac{1}{11T}}\right)\right]$$

To find J(t), take inverse Laplace Transform and using $L^{-1}\left[\frac{1}{s}\right] = 1$ and

$L^{-1}\left[\frac{1}{s+a}\right] = e^{-as}$, we have

$$J(t) = \frac{1}{q}\left[\frac{1}{2}\left(1 - e^{-t/T}\right) + \left(1 - e^{-t/4T}\right) + \frac{2}{3}\left(1 - e^{-t/11T}\right)\right]$$

4.6 Summary

We have studied about Standard Linear Solid Model and Generalised viscoelastic models namely Maxwell and Kelvin. The constitutive equations of these models have been derived. We have also studied about creep and relaxation phenomena.

4.7 Keywords: Standard Linear Solid, Generalised Kelvin model, Generalised Maxwell model, creep phase, relaxation phase

4.8 Self-assessment Questions

Q1. Describe the Standard Linear Solid model of viscoelasticity. Find its constitutive equation. Also discuss its creep phase and relaxation phase.

Q2. Describe the Generalised Kelvin model of viscoelasticity. Find its constitutive equation and hence discuss the creep phase test.

Q3. Describe the Generalised Maxwell model of viscoelasticity. Find its constitutive equation and hence discuss the relaxation phase.

4.9 Suggested Readings

1. Y.C. Fung, Foundations of Solid Mechanics, Prentice Hall, New Delhi.
2. W. Flugge, Viscoelasticity, Springer Verlag.
3. R.M. Christensen, Theory of Viscoelasticity- An Introduction, 2nd Edition, 1982, Academic Press Inc., New York.
4. D.R. Bland, The Theory of Linear Viscoelasticity, Pergamon, New York, 1960.

Chapter 5

Correspondence Principle of linear viscoelasticity and its applications

5.1 Objectives

In this chapter, we shall discuss about Correspondence Principle of linear viscoelasticity and its applications to the deformation of a viscoelastic thick-walled tube in plane strain.

5.2 Introduction

In this chapter, some simple stress problems involving a viscoelastic material have been considered and solved. The general problem is the same for elastic and viscoelastic structures. In both cases, the three basic sets of equations must be satisfied: the equilibrium equations, the kinematic relations, and the constitutive equations of the material. The first two of these are common to elastic and viscoelastic materials. The only difference between elastic and viscoelastic materials is in the constitutive equations of the material. For viscoelastic materials, Hooke's law is to be replaced by another equation. Thus the solution of viscoelastic problem can be obtained with the help of corresponding solution of elastic problem. If the solution of an elastic problem is known, the Laplace transformed solution of the corresponding viscoelastic problem can be obtained on replacing elastic moduli μ and K by the corresponding transformed moduli $\overline{\mu}$ and $\overline{K}$ respectively and the actual load by their Laplace transform. This is known as Correspondence Principle of linear viscoelasticity.

5.3 Correspondence Principle of linear viscoelasticity

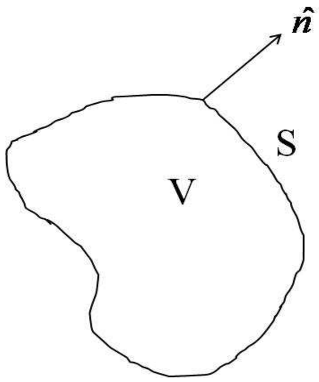
We know that the stress-strain relation for 1-D viscoelastic material is

$$\sum_r p_r \frac{\partial^r \tau}{\partial t^r} = \sum_r q_r \frac{\partial^r e}{\partial t^r}$$

Or $P(\tau) = Q(e)$

where $P = \sum_r p_r \frac{\partial^r \tau}{\partial t^r}$, $Q = \sum_r q_r \frac{\partial^r e}{\partial t^r}$

Generalization: Consider a stress problem, let a body consisting of volume V bounded by the surface S.



The basic equations are

1. Equations of Equilibrium: $\tau_{ij,j} + F_i = 0$ (1)

2. Kinematic Relations: $e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$ (2)

3. Boundary conditions are:

$$\left.\begin{aligned} &T_i^{\hat{n}} = \tau_{ij}\eta_j = f_i \quad \text{on} \quad S \\ &\text{or} \quad u_i = \phi_i \quad \text{on} \quad S \end{aligned}\right\} \quad (3)$$

where f_i and ϕ_i are prescribed functions.

4. Constitutive equations: For an elastic material $\tau_{ij} = \lambda\delta_{ij}\,\vartheta + 2\mu\, e_{ij}$ (4)

We define, Deviatoric strain ε_{ij} and Deviatoric stress p_{ij} where

$$\varepsilon_{ij} = e_{ij} - \frac{1}{3}\vartheta\delta_{ij} \qquad (\vartheta = e_{ii}) \tag{5}$$

$$p_{ij} = \tau_{ij} - \frac{1}{3}\theta\delta_{ij} \qquad (\theta = \tau_{ii}) \tag{6}$$

Using equation (5) and (6) in equation (4), we get

$$p_{ij} + \frac{1}{3}\theta\,\delta_{ij} = \lambda\delta_{ij}\,\vartheta + 2\mu\left[\varepsilon_{ij} + \frac{1}{3}\vartheta\delta_{ij}\right]$$

$$\Rightarrow p_{ij} + \frac{1}{3}\theta\,\delta_{ij} = \left[\lambda + \frac{2}{3}\mu\right]\delta_{ij}\,\vartheta + 2\mu\,\varepsilon_{ij} = K\delta_{ij}\,\vartheta + 2\mu\,\varepsilon_{ij} \tag{7}$$

where $\mathrm{K} = \lambda + \frac{2}{3}\mu$ = Bulk Modulus

Taking i = j in equation (4), we have

$$\tau_{ij} = \lambda\delta_{ij}\,\vartheta + 2\mu\, e_{ij} = (3\lambda + 2\mu)\vartheta = 3\left(\lambda + \frac{2}{3}\mu\right)\vartheta$$

$$\Rightarrow \theta = 3K\vartheta \tag{8}$$

Using (8) in equation (7), we have

$$p_{ij} + \frac{1}{3}\left(3K\vartheta\delta_{ij}\right) = K\delta_{ij}\vartheta + 2\mu\varepsilon_{ij}$$

$$\Rightarrow p_{ij} = 2\mu\varepsilon_{ij} \tag{9}$$

Equation (1)-(3) and (8)-(9) hold for an elastic material. For a viscoelastic material, equation (1) holds for a continuous material elastic or viscoelastic.

We write the viscoelastic stress-strain relation for 3-D viscoelastic material:

$$\sum_r p'_r \frac{\partial^r p_{ij}}{\partial t^r} = \sum_r q'_r \frac{\partial^r \varepsilon_{ij}}{\partial t^r}$$

$$\text{and} \quad \sum_r p''_r \frac{\partial^r \theta}{\partial t^r} = \sum_r q''_r \frac{\partial^r \vartheta}{\partial t^r} \qquad (10)$$

$$P'(p_{ij}) = Q'(\varepsilon_{ij})$$

$$\text{and} \quad P''(\theta) = Q''(\vartheta) \qquad (11)$$

where

$$P' = \sum_{r=1}^{m'} p'_r \frac{\partial^r}{\partial t^r} \quad , \quad Q' = \sum_{r=1}^{n'} q'_r \frac{\partial^r}{\partial t^r}$$

$$\text{and} \quad P'' = \sum_{r=1}^{m''} p''_r \frac{\partial^r}{\partial t^r} \quad , \quad Q'' = \sum_{r=1}^{n''} q''_r \frac{\partial^r}{\partial t^r}$$

Remark: Equation (9) and (10a) is for deviatoric changes of an elastic and viscoelastic material and equation (8) and (10b) is for the dilatational changes of an elastic and viscoelastic material respectively.

Correspondence Principle:

Consider a continuous material under constant load. For an elastic body, nothing depends upon time. But for a viscoelastic material, $\phi_{ij}, \varepsilon_{ij}, \theta, \vartheta, f_i, \phi_i$ depends upon time.

We use the method of Laplace transforms.

We take L.T. of equations (1), (3) and (11), we obtain

$$\bar{\tau}_{ij,j} + \bar{F}_i = 0 \quad \text{in} \quad V \tag{12}$$

$$\begin{aligned} &\bar{\tau}_{ij}\eta_j = \bar{f}_i \quad \text{on} \quad S \\ &\text{or} \quad \bar{u}_i = \bar{\phi}_i \quad \text{on} \quad S \end{aligned} \tag{13}$$

$$\begin{aligned} &P'(s)\bar{p}_{ij} = Q'(s)\bar{\varepsilon}_{ij} \\ &\text{and} \quad P''(s)\bar{\theta} = Q''(s)\bar{\vartheta} \end{aligned} \tag{14}$$

where

$$\begin{aligned} &P'(s) = \sum_{r=1}^{m'} p'_r s^r \quad , \quad Q'(s) = \sum_{r=1}^{n'} q'_r s^r \\ &\text{and} \quad P''(s) = \sum_{r=1}^{m''} p''_r s^r \quad , \quad Q''(s) = \sum_{r=1}^{n''} q''_r s^r \end{aligned} \tag{15}$$

Assuming that there is no deformation at t=0.

We define Transform shear modulus μ^* and Transform bulk modulus K^* by the relation

$$\begin{aligned} &2\mu^*(s) = \frac{Q'(s)}{P'(s)} \\ &\text{and} \quad 3K^*(s) = \frac{Q''(s)}{P''(s)} \end{aligned} \tag{16}$$

Then equation (14) becomes

$$\begin{aligned} &\bar{p}_{ij} = 2\mu^*(s)\bar{\varepsilon}_{ij} \\ &\text{and} \quad \bar{\theta} = 3K^*(s)\bar{\vartheta} \end{aligned} \tag{17}$$

Equations (12), (13) and (17) hold for a viscoelastic material.

On comparing the two sets of equations for an elastic material and viscoelastic material, we observe that the quantities, $p_{ij}, \varepsilon_{ij}, \theta, \vartheta, f_i, \phi_i$ are replaced by $\bar{p}_{ij}, \bar{\varepsilon}_{ij}, \bar{\theta}, \bar{\vartheta}, \bar{f}_i, \bar{\phi}_i$ and μ and K are replaced by μ^* and K^* respectively.

Hence, we have the following Correspondence Principle of Linear Viscoelasticity:

"If we know the solution of any problem for an elastic material, then the Transform of solution of corresponding viscoelastic material can be known by replacing the quantities $p_{ij}, \varepsilon_{ij}, \theta, \vartheta, f_i, \phi_i$ by their Laplace transforms and the elastic constants μ and K are replaced by μ^* and K^* respectively."

5.4 Applications of Correspondence Principle

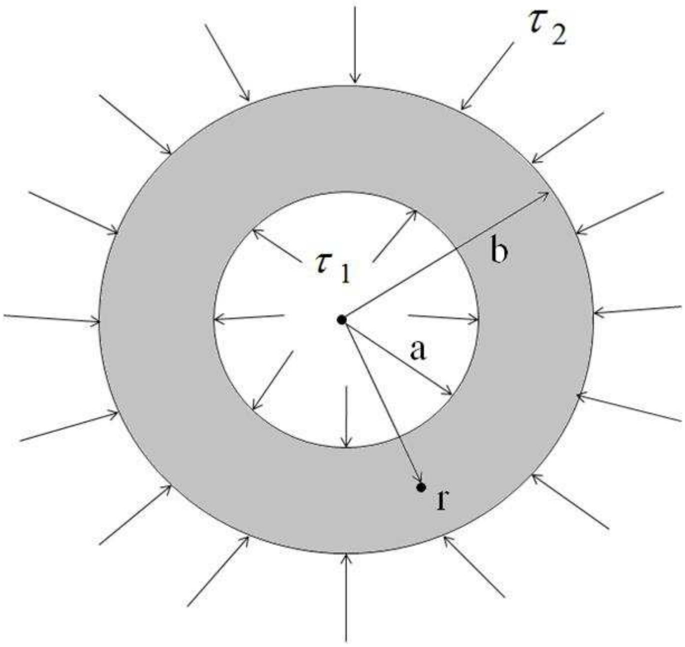
Problem – I: Deformation of long thick walled tube due to internal pressure τ_1 and external pressure τ_2.

Problem – II: Deformation of thick walled tube under internal pressure τ_1 and tube is in contact with a rigid medium.

Problem – I

Consider a long thick walled tube of inner radius 'a' and external radius 'b' under no external forces.

Let there be internal pressure τ_1 and internal pressure τ_2 on the tube.



Since the tube is ring, it is plane strain problem.

Let the axis of the tube is taken along z-axis. We choose the x_1x_2 plane.

Choosing the cylindrical co-ordinate system (r, θ, z).

Due to axial symmetry, $\frac{\partial}{\partial \theta} \equiv 0$

Therefore

$$u_r = u_r(r) \ , u_\theta = 0 \ , \ u_z = 0 \tag{1}$$

Therefore equation of equilibrium gives

$$(\lambda + \mu) grad\ divu + \mu \nabla^2 u + F = 0$$

$$(\lambda + 2\mu) grad\ divu - \mu\, curlcurlu = 0$$

By equation (1) $curlcurlu = 0$

So, equation of equilibrium gives $grad \;\; div\, u = 0$

$$\Rightarrow \frac{d}{dr}\left(\frac{du}{dr}+\frac{u}{r}\right)=0$$

Integrating, we have $\left(\frac{du}{dr}+\frac{u}{r}\right)=2A$ (A is constant)

$$\Rightarrow \left(r\frac{du}{dr}+u\right)=2Ar \quad \Rightarrow ru = Ar^2 + B$$

where B is constant of integration.

$$\Rightarrow u = Ar + \frac{B}{r} \tag{2}$$

Boundary conditions are:

$$\begin{aligned} &\tau_{rr} = -\tau_1 \text{ , when } \quad r = a \\ &\text{and } \; \tau_{rr} = -\tau_2 \text{ , when } \quad r = b \end{aligned} \tag{3}$$

From equation (2), the stresses are given by

$$\tau_{ij} = \lambda\delta_{ij}\vartheta + 2\mu e_{ij}$$

$$\begin{aligned} \Rightarrow \tau_{rr} &= \lambda\, div\, u + 2\mu\, e_{rr} \\ &= \lambda\left(\frac{du}{dr}+\frac{u}{r}\right)+2\mu\frac{\partial u}{\partial r} \\ &= 2\lambda A + 2\mu\left(A - \frac{B}{r^2}\right) \end{aligned}$$

$$\Rightarrow \tau_{rr} = 2(\lambda+\mu)A + 2\mu\left(-\frac{B}{r^2}\right) \tag{4}$$

and $\tau_{\theta\theta} = \lambda\, div\, u + 2\mu\, e_{\theta\theta}$

$$\Rightarrow \tau_{\theta\theta} = 2(\lambda+\mu)A + 2\mu\left(\frac{B}{r^2}\right) \qquad (5)$$

and $\tau_{zz} = \sigma(\tau_{rr} + \tau_{\theta\theta})$

$\tau_{r\theta} = 0$

From equations (3) and (4), solving for A, B

$$2(\lambda+\mu)A + 2\mu\left(-\frac{B}{a^2}\right) = -\tau_1$$

and $2(\lambda+\mu)A + 2\mu\left(-\frac{B}{b^2}\right) = -\tau_2$

We get,

$$A = \frac{\tau_1 a^2 - \tau_2 b^2}{2(\lambda+\mu)(b^2-a^2)}, \qquad B = \frac{(\tau_1-\tau_2)b^2a^2}{2\mu(b^2-a^2)}$$

or $$A = \frac{\tau_1 a^2 - \tau_2 b^2}{2\left(K+\frac{\mu}{3}\right)(b^2-a^2)}, \qquad B = \frac{(\tau_1-\tau_2)b^2a^2}{2\mu(b^2-a^2)} \qquad \left(\because K = \left(\lambda + \frac{2\mu}{3}\right)\right)$$

Using in equations (2), (4) and (5), the elastic solution is

$$\Rightarrow u = \frac{\tau_1 a^2 - \tau_2 b^2}{2\left(K+\frac{\mu}{3}\right)(b^2-a^2)} r + \frac{(\tau_1-\tau_2)b^2a^2}{2\mu(b^2-a^2)}\frac{1}{r} \qquad (6)$$

$$\Rightarrow \tau_{rr} = \frac{\tau_1 a^2 - \tau_2 b^2}{(b^2-a^2)} - \frac{(\tau_1-\tau_2)b^2a^2}{(b^2-a^2)r^2}$$

(7)

and $$\tau_{\theta\theta} = \frac{\tau_1 a^2 - \tau_2 b^2}{(b^2-a^2)} + \frac{(\tau_1-\tau_2)b^2a^2}{(b^2-a^2)r^2}$$

To obtain the viscoelastic solution, we apply the correspondence principle which states that the quantities $\tau_1, \tau_2,\ u, \tau_{rr}, \tau_{\theta\theta}$ must be replaced by their Laplace transform and the elastic constants μ and K by $\mu^*(s) = \frac{Q'(s)}{2P'(s)}$ and $K^*(s) = \frac{Q''(s)}{3P''(s)}$.

We also assume that there is step loading.

Therefore

$\tau_1(t) = \tau_1 H(t)$, $\tau_2(t) = \tau_2 H(t)$ where H(t) is unit step function.

Taking the Laplace transform, we have

$$\bar{\tau}_1 = \frac{\tau_1}{s}, \qquad \bar{\tau}_2 = \frac{\tau_2}{s} \tag{8}$$

Also equation (7) is free from elastic constants. Therefore stresses do not change for the viscoelastic material. Hence, we calculate only u.

From equation (6) and (8), the Laplace transform of viscoelastic solution is

$$\Rightarrow \bar{u}(s) = \frac{3P'P''}{s(2Q''P' + Q'P'')}\left[\frac{\tau_1 a^2 - \tau_2 b^2}{(b^2 - a^2)}\right] r + \frac{P'}{Q's}\left[\frac{(\tau_1 - \tau_2) b^2 a^2}{(b^2 - a^2)}\right]\frac{1}{r}$$

We now choose the specific material to obtain the values of P', Q', P'', Q''. We consider two cases.

Case I:

We assume that the material is elastic in dilatation and Kelvin behaviour in distortion.

Since material is elastic in dilatation so, $\theta = 3K\vartheta$

L.T. gives us, $\bar{\theta} = 3K\bar{\vartheta}$

On comparing with $P''(s)\bar{\theta} = Q''(s)\bar{\vartheta}$, we get

$P''(s) = 1, Q''(s) = 3K$

Since material is Kelvin behaviour in distortion $\tau = q_0 e + q_1 \dot{e}$

$$\Rightarrow P'(t) = 1 \ , \quad Q'(t) = q_0 + q_1 \frac{\partial}{\partial t} \qquad (11)$$
$$P'(s) = 1 \quad , \quad Q'(s) = q_0 + q_1 s$$

Using equations (10) and (11) in equations (9), we get

$$\bar{u}(s) = \frac{3}{s(q_0 + 6K + q_1 s)}\left[\frac{\tau_1 a^2 - \tau_2 b^2}{(b^2 - a^2)}\right] r + \frac{1}{(q_0 + q_1 s)s}\left[\frac{(\tau_1 - \tau_2) b^2 a^2}{(b^2 - a^2)}\right]\frac{1}{r}$$

The solution is in Laplace transform domain.

Taking Inverse Laplace transform and using $\left(\frac{1}{s(s+a)} = \frac{1}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right]\right)$, we get

$$\bar{u}(s) = \frac{1}{q_1} \times \frac{3q_1}{q_0 + 6K}\left[\frac{\tau_1 a^2 - \tau_2 b^2}{(b^2 - a^2)}\right]\left[\frac{1}{s} - \frac{1}{s + \frac{q_0 + 6K}{q_1}}\right] r + \frac{1}{q_1} \times \frac{q_1}{q_0}\left[\frac{(\tau_1 - \tau_2) b^2 a^2}{(b^2 - a^2)}\right]\left[\frac{1}{s} - \frac{1}{s + \frac{q_0}{q_1}}\right]\frac{1}{r}$$

Taking Inverse L.T., we obtain

$$u(t) = \frac{3}{q_0 + 6K}\left[\frac{\tau_1 a^2 - \tau_2 b^2}{(b^2 - a^2)}\right]\left[1 - e^{-\frac{q_0 + 6K}{q_1}t}\right] r + \frac{1}{q_0}\left[\frac{(\tau_1 - \tau_2) b^2 a^2}{(b^2 - a^2)}\right]\left[1 - e^{-\frac{q_0}{q_1}t}\right]\frac{1}{r} \qquad (12)$$

is required displacement for Kelvin model.

Case II:

We assume that the material is elastic in dilatation and Maxwell viscoelastic behaviour in distortion.

Since material is elastic in dilatation so, $\theta = 3K\vartheta$

L.T. gives us, $\bar{\theta} = 3K\bar{\vartheta}$

On comparing with $P''(s)\bar{\theta} = Q''(s)\bar{\vartheta}$, we get

$$P''(s) = 1 \ , \quad Q''(s) = 3K \tag{13}$$

Since material is Maxwell behaviour in distortion, the stress-strain relation is

$$\tau + p_1\dot{\tau} = q_1\dot{e}$$

Taking L.T., we obtain

$$(1 + p_1 s)\bar{\tau} = q_1 s\bar{e}(s)$$

Comparing with $P'(s)\bar{\tau} = Q'(s)\bar{e}$, we get

$$P'(s) = (1 + sp_1), \qquad Q'(s) = q_1(s) \tag{14}$$

We also assume that there is step loading.

Therefore

$$\tau_1(t) = \tau_1 H(t) \quad , \qquad \tau_2(t) = \tau_2 H(t)$$

Taking the Laplace transform, we have

$$\bar{\tau}_1 = \frac{\tau_1}{s}, \bar{\tau}_2 = \frac{\tau_2}{s} \tag{15}$$

Using in equation (9)

$$\bar{u}(s)=\frac{3(1+p_1 s)}{s((1+p_1 s)6K+q_1 s)}\left[\frac{\tau_1 a^2-\tau_2 b^2}{\left(b^2-a^2\right)}\right]r+\frac{(1+p_1 s)}{(q_1)s^2}\left[\frac{(\tau_1-\tau_2)b^2 a^2}{\left(b^2-a^2\right)}\right]\frac{1}{r}$$

Taking Inverse L.T., we obtain

$$u(t)=\frac{1}{2K}\left[\frac{\tau_1 a^2-\tau_2 b^2}{\left(b^2-a^2\right)}\right]\left[1-\frac{q_1}{q_1+6Kp_1}e^{-\frac{6K}{q_1+6Kp_1}t}\right]r+\frac{1}{q_1}\left[\frac{(\tau_1-\tau_2)b^2 a^2}{\left(b^2-a^2\right)}\right][t+p_1]\frac{1}{r}$$

is required solution.

Particular case:

When the outer surface of the tube is free from external pressure.

Take $\tau_2=0$

Problem-II

Consider the thick walled tube subjected to the internal pressure and τ_1 outer surface is n contact with rigid medium. Like last article, we have

$$u_r=u_r(r)\,,\quad u_\theta=0\,,\quad u_z=0 \tag{1}$$

Boundary conditions are:

$$\begin{aligned}&\tau_{rr}=-\tau\,,\ \text{when}\quad r=a\\&\text{and}\,u_r=u=0\,,\ \text{when}\quad r=b\end{aligned} \tag{2}$$

Therefore equation of equilibrium gives

$$(\lambda+\mu)grad\ divu+\mu\nabla^2 u+F=0$$

$$(\lambda+2\mu)grad\ divu-\mu\, curlcurlu=0$$

By equation (1), $curlcurlu=0$

So, equation of equilibrium gives

$$grad\ \ div\ u = 0$$

$$\Rightarrow \frac{d}{dr}\left(\frac{du}{dr} + \frac{u}{r}\right) = 0$$

Integrating, we have

$$\left(\frac{du}{dr} + \frac{u}{r}\right) = 2A \qquad \text{(A is constant)}$$

$$\Rightarrow \left(r\frac{du}{dr} + u\right) = 2Ar$$

$$\Rightarrow ru = Ar^2 + B$$

where B is constant of integration.

$$\Rightarrow u = Ar + \frac{B}{r} \tag{3}$$

The stresses, corresponding to equation (3) are

$$\left.\begin{aligned} \tau_{rr} &= 2(\lambda + \mu)A + 2\mu\left(-\frac{B}{r^2}\right) \\ \tau_{\theta\theta} &= 2(\lambda + \mu)A + 2\mu\left(\frac{B}{r^2}\right) \end{aligned}\right\} \tag{4}$$

From equations (2), (3) and (4), solving for A, B we get

$$2(\lambda + \mu)A + 2\mu\left(-\frac{B}{a^2}\right) = -\tau$$

and $Ab + \dfrac{B}{b} = 0$

We get,

$$A = \frac{-\tau a^2}{2\left[(\lambda+\mu)a^2 + \mu b^2\right]},\ \ B = \frac{\tau b^2 a^2}{2\left[(\lambda+\mu)a^2 + \mu b^2\right]}$$

Using in equations (3) and (4), the elastic solution is

$$u = \frac{\tau a^2}{2\left(Ka^2 + \left(\frac{a^2}{3} + b^2\right)\mu\right)}\left[-r + \frac{b^2}{r}\right] \tag{5}$$

$$\tau_{rr} = \frac{-\tau a^2}{\left(Ka^2 + \left(\frac{a^2}{3} + b^2\right)\mu\right)}\left[K + \left(\frac{1}{3} + \frac{b^2}{r^2}\right)\mu\right] \tag{6}$$

To obtain the viscoelastic solution, we apply the correspondence principle.

The Laplace transform of viscoelastic solution is

$$\bar{u}(s) = \frac{\bar{\tau} a^2}{2\left(K^* a^2 + \left(\frac{a^2}{3} + b^2\right)\mu^*\right)}\left[-r + \frac{b^2}{r}\right] \tag{7}$$

$$\bar{\tau}_{rr} = \frac{-\bar{\tau} a^2}{\left(K^* a^2 + \left(\frac{a^2}{3} + b^2\right)\mu^*\right)}\left[K^* + \left(\frac{1}{3} + \frac{b^2}{r^2}\right)\mu^*\right] \tag{8}$$

We also assume that there is step loading.

Therefore

$$\tau(t) = \tau H(t)$$

Taking the Laplace transform, we have

$$\bar{\tau}(s) = \frac{\tau}{s} \tag{9}$$

$$\mu^*(s) = \frac{Q'(s)}{2P'(s)} \quad \text{and} \qquad K^*(s) = \frac{Q''(s)}{3P''(s)}.$$

We assume that the material is elastic in dilatation and Maxwell viscoelastic behaviour in distortion. (Similarly we can discuss about elastic in dilatation and Kelvin viscoelastic behaviour in distortion.)

Since material is elastic in dilatation so, $\theta = 3K\vartheta$

L.T. gives us, $\overline{\theta} = 3K\overline{\vartheta}$

On comparing with $P''(s)\overline{\theta} = Q''(s)\,\overline{\vartheta}$, we get

$$P''(s) = 1\,,\ Q''(s) = 3K \quad \Rightarrow \qquad K^* = K$$

Since material is Maxwell behaviour in distortion, the stress-strain relation is

$$\tau + p_1\dot{\tau} = q_1\dot{e}$$

Taking L.T., we obtain

$$(1 + p_1 s)\overline{\tau} = q_1\, s\,\overline{e}(s)$$

Comparing with $P'(s)\overline{\tau} = Q'(s)\overline{e}$, we get,

$$P'(s) = (1 + sp_1), \qquad Q'(s) = q_1 s$$

Using these values in equations (7) and (8), we get

$$\overline{u}(s) = \frac{3\tau(1 + sp_1)}{s(6K + \alpha s)}\left[-r + \frac{b^2}{r}\right] \qquad \text{where} \qquad \alpha = 6Kp_1 + q_1\left(1 + \frac{3b^2}{a^2}\right)$$

$$\bar{\tau}_{rr}(s) = \frac{-\tau a^2}{s\left(Ka^2 + \left(\frac{a^2}{3} + b^2\right)\frac{q_1 s}{2(1+p_1 s)}\right)}\left[K + \left(\frac{1}{3} + \frac{b^2}{r^2}\right)\frac{q_1 s}{2(1+p_1 s)}\right]$$

$$\bar{\tau}_{rr}(s) = -\tau\left[\frac{1}{s} - \frac{3b^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right)q_1}{6K + \alpha s}\right]$$

Taking Inverse L.T., we obtain

$$u(t) = \frac{\tau}{2K}\left[1 - \left(1 + \frac{3b^2}{a^2}\right)\frac{q_1}{\alpha}e^{-\frac{6K}{\alpha}t}\right]\left[-r + \frac{b^2}{r}\right]$$

$$\tau_{rr}(t) = -\tau\left[1 - 3b^2\left(\frac{1}{a^2} - \frac{1}{r^2}\right)\frac{q_1}{\alpha}e^{-\frac{6K}{\alpha}t}\right]$$

$$\tau_{\theta\theta}(t) = -\tau\left[1 - 3b^2\left(\frac{1}{a^2} + \frac{1}{r^2}\right)\frac{q_1}{\alpha}e^{-\frac{6K}{\alpha}t}\right]$$

which is required solution.

Problem III

Outer surface is in contact with rigid medium and inner surface is acted on by pressure τ_1.

Boundary conditions are:

$u = 0$ at external

$\tau_{rr} = -\tau_1$ at inner boundary.

5.5 Summary

We have studied about the correspondence principle of linear viscoelasticity and its applications to two-dimensional problems.

5.6 Keywords: Viscoelasticity, Correspondence principle, thick walled tube, internal pressure, External pressure, axial symmetry, elastic material.

5.7 Self-assessment Questions

Q 1. State and prove general correspondence principle of viscoelasticity.

Q 2. Describe deformation of long thick walled tube due to internal pressure τ_1 and external pressure τ_2.

Q 3. Describe deformation of thick walled tube under internal pressure τ_1 and tube is in contact with a rigid medium.

Q 4. Describe deformation of thick walled tube when outer surface is in contact with rigid medium and inner surface is acted on by pressure τ_1 for the material elastic in dilatation and Kelvin viscoelastic behaviour in distortion.

Q 5. Describe deformation of thick walled tube when outer surface is in contact with rigid medium and inner surface is acted on by pressure τ_1 for the material elastic in dilatation and Standard Linear Solid viscoelastic behaviour in distortion.

5.8 Suggested Readings

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2. W. Flugge, Viscoelasticity, Springer Verlag.

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Chapter-6

Fundamental Equations of Elastodynamics and Seismic Waves

6.1 Objectives

In this chapter, we shall discuss about Field equations of Elastodynamics, propagation of waves in an isotropic elastic solid medium, waves of dilatation and distortion. We also discuss about elastic and plane elastic waves.

6.2 Introduction

The differential equations of motion of an elastic solid can be obtained at once from the equations of equilibrium [$\tau_{ji,j} + F_i = 0$, $(i, j = 1, 2, 3)$] by invoking the principle of D' Alembert and adding the forces of inertia to the components of the body force. If $\rho(x_1, x_2, x_3)$ is the density of the medium, then the components of the force of inertia acting on the mass contained within the volume element $d\tau$ are $-\rho \dfrac{\partial^2 u_i}{\partial t^2}\, d\tau$.

Hence adding to the components F_i of the body force **F** in equilibrium equations, the components of the force of inertia per unit volume gives the system of equations

$$\tau_{ji,j} + F_i = \rho\, \ddot{u}_i$$

where $\dfrac{\partial^2 u_i}{\partial t^2} \equiv \ddot{u}_i$,

which are known as equations of motion.

6.3 Field equations of Elastodynamics

We know that

I Equations of Motion:

$$\tau_{ji,j} + \rho f_i = \rho \ddot{u}_i$$

where

τ_{ji} is stress tensor

$$\tau_{ji,j} = \frac{\partial}{\partial x_j}\left(\tau_{ji}\right)$$

f_i are Body force per unit mass

$\ddot{u}_i = \dfrac{\partial^2 u_i}{\partial t^2}$ is Acceleration

$\rho \ddot{u}_i$ is known Inertia term

II Constitutive equations (Generalised Hook's Law):

For isotropic homogenous elastic medium, the stress-strain relations are given by

$$\tau_{ij} = \lambda \vartheta \delta_{ij} + 2\mu e_{ij},$$

where λ, μ are Lame's Constants

$\vartheta = e_{kk} = e_{11} + e_{22} + e_{33}$ is known as Dilatation or Cubical dilatation.

e_{ij} is Strain Tensor

τ_{ij} is Stress Tensor

III Strain-displacement relations:

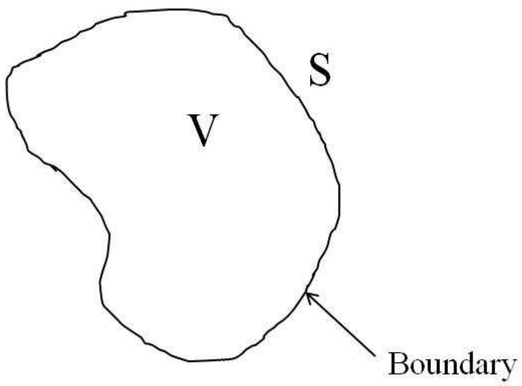
$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

IV Boundary conditions

Either displacements or tractions (stresses) are prescribed on boundary, i.e.,

$u_i = \phi_i$ or $t_i = \psi_i$ are known.

In case of mixed BVP, the displacements are given on some portion and stresses are given on remaining portion, i.e.



On boundary

$S = S_u + S_T$

6.4 Waves

Wave is a disturbance travelling through a medium without producing a permanent displacement to the medium such that the energy is propagated to distinct points. Waves in air, in liquid, in the light, in the string, in the electric cable are very common. Examples are: (i) when a pebble is dropped into a pond, water waves travel radially outwards; (ii) when a piano is played, the wires vibrate and sound waves spread through the room. Also the conversation we make is carried by sound waves; (iii) the objects we see are made visible by light waves.

Definition: A wave may be defined as a disturbance that propagates with a constant velocity without change of shape.

6.5 Field equations of Elastic waves

Equations of motion are:

$$\tau_{ji,j} + \rho f_i = \rho \ddot{u}_i \tag{1}$$

Stress-strain relations are

$$\tau_{ji} = \lambda \vartheta \delta_{ij} + \mu\left(u_{i,j} + u_{j,i}\right) \tag{2}$$

where λ , μ are elastic constants.

Then, using (2) in (1), we get

$$\lambda \delta_{ij} \vartheta_{ij} + \mu\left(u_{i,jj} + u_{j,ij}\right) + \rho f_i = \rho \ddot{u}_i$$

$$\Rightarrow \lambda \vartheta_{,i} + \mu\left(\nabla^2 u_i + \vartheta_{,i}\right) + \rho f_i = \rho \ddot{u}_i$$

$$\Rightarrow (\lambda + \mu)\vartheta_{,i} + \mu \nabla^2 u_i + \rho f_i = \rho \ddot{u}_i$$

These are known as Navier's equations of motion. These are the 3 scalar equations.

These are the displacement equations of motion.

Navier's equations of motion in Vector form:-

If u_i are the displacement components, then

$$\vec{u} = u_i \vec{e}_i$$

where $\vec{e}_i$ are the unit vectors.

So we have

$$(\lambda + \mu)\vec{e}_i \vartheta_{,i} + \mu \vec{e}_i \nabla^2 u_i + \rho \vec{e}_i f_i = \rho \vec{e}_i \ddot{u}_i$$

$$\Rightarrow (\lambda + \mu) grad\, \vartheta + \mu \nabla^2 \vec{u} + \rho \vec{f} = \rho \ddot{\vec{u}} \tag{3}$$

These are the Navier's equation of motion in vector form.

Since $\vartheta = div\,\vec{u}$, then

$$(\lambda+\mu)grad\ div\ \vec{u}+\mu\nabla^2\vec{u}+\rho\,\vec{f}=\rho\ddot{\vec{u}}$$

Also

$$\nabla^2\vec{u}=grad\ div\,\vec{u}-curl\,curl\,\vec{u}$$

Then above equation becomes

$$(\lambda+2\mu)\ grad\ div\ \ \vec{u}-\mu\,curl\,curl\ \vec{u}+\rho\,\vec{f}=\rho\ddot{\vec{u}}$$

or

$$(\lambda+2\mu)\nabla\nabla.\vec{u}-\mu\nabla\times\nabla\times\vec{u}+\rho\,\vec{f}=\rho\ddot{\vec{u}}$$

These are the system of coupled partial differential equations for displacement vector.

6.6 Waves in an isotropic elastic solid medium:

In the absence of body force, the equations of motion (3) become

$$(\lambda+\mu)grad\,\vartheta+\mu\nabla^2\vec{u}=\rho\ddot{\vec{u}} \tag{4}$$

Taking divergence of both sides, we obtain

$$(\lambda+\mu)div\,grad\,\vartheta+\mu\,div\left(\nabla^2\vec{u}\right)=\rho\frac{\partial^2}{\partial t^2}(div\,\vec{u})$$

$$\Rightarrow(\lambda+\mu)\nabla^2\vartheta+\mu\nabla^2(div\,\vec{u})=\rho\frac{\partial^2}{\partial t^2}(div\,\vec{u})$$

$$\Rightarrow(\lambda+\mu)\nabla^2\vartheta+\mu\nabla^2\vartheta=\rho\frac{\partial^2\vartheta}{\partial t^2}$$

$$\Rightarrow (\lambda + 2\mu)\nabla^2 \vartheta = \rho \ddot{\vartheta}$$

$$\Rightarrow (\lambda + 2\mu)\nabla^2 \vartheta = \rho \frac{\partial^2 \vartheta}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vartheta = \left(\frac{\rho}{\lambda + 2\mu} \right) \frac{\partial^2 \vartheta}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vartheta = \frac{1}{\alpha^2} \frac{\partial^2 \vartheta}{\partial t^2} \qquad (5)$$

where

$$\alpha^2 = \frac{\lambda + 2\mu}{\rho}$$

Since

$$K = \lambda + \frac{2\mu}{3}$$

$$\Rightarrow \alpha^2 = \frac{\lambda + 2\mu}{\rho} = \frac{K + \frac{4}{3}\mu}{\rho} \qquad (6)$$

From (5), $div\,\vec{u} = \vartheta$ satisfies 3D (scalar) wave equation with velocity α.

Next, take curl of both sides of equation (4), we obtain

$$(\lambda + \mu) curl\; grad\, \vartheta + \mu\, curl\left(\nabla^2 \vec{u}\right) = \rho\, curl\, \ddot{\vec{u}}$$

$$\Rightarrow \mu \nabla^2 (curl\, \vec{u}) = \rho \frac{\partial^2}{\partial t^2} (curl\, \vec{u})$$

Let us define

$$\vec{\Omega} = curl\,\vec{u} = \nabla \times \vec{u}$$

$$\Rightarrow \mu \nabla^2 \vec{\Omega} = \rho \frac{\partial^2 \vec{\Omega}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vec{\Omega} = \frac{\rho}{\mu} \frac{\partial^2 \vec{\Omega}}{\partial t^2}$$

$$\Rightarrow \nabla^2 \vec{\Omega} = \frac{1}{\beta^2} \frac{\partial^2 \vec{\Omega}}{\partial t^2} \quad , \qquad (7)$$

where $\beta^2 = \frac{\mu}{\rho}$

Equation (7) is a vector wave equation with velocity β.

Equation (5) shows that dilatational disturbance ϑ may be transmitted through an elastic medium with velocity α. And equation (7) shows that a rotational disturbance may be transmitted through an elastic medium with velocity β.

We therefore, conclude that the disturbance in an infinite homogeneous isotropic, elastic medium can be propagated in the form of two types of waves:

1. Dilatational waves with velocity of propagation α.
2. Rotational waves with velocity of propagation β.

If $\lambda = \mu$ (Poisson's case)

Then $\alpha^2 = \frac{3\mu}{\rho}$, $\beta^2 = \frac{\mu}{\rho}$

$$\Rightarrow \alpha = \sqrt{3}\beta \ \Rightarrow \ \alpha > \beta$$

Therefore Dilatation waves moves faster than rotational waves.

Therefore, dilatational waves arrive first while rotational waves arrive after that on a seismogram. For this reason, dilatational waves are also called primary waves and rotational waves are called secondary waves.

6.7 Seismic wave potentials:

The displacement $\vec{u}$ is also expressed in terms of P-wave scalar potential ϕ and S-wave vector potential $\vec{\psi}$, using the Helmholtz's decomposition theorem,

$$\vec{u} = \nabla\phi + curl\,\vec{\psi}, \qquad div\,\vec{\psi} = 0$$

in equation of motion, we can get

$$\nabla^2\phi = \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2} \quad \text{for P-wave}$$

$$\nabla^2\vec{\psi} = \frac{1}{\beta^2}\frac{\partial^2\vec{\psi}}{\partial t^2} \quad \text{for S-wave.}$$

Dilatational waves are also called irrotational waves or P-waves.

Since $div\ \vec{\psi} = 0$, it follows that a rotational wave is free of expansion or compression of volume. For this reason, the rotational wave is also called equivoluminal or dilatationless or secondary waves or S-wave.

Remark 1: The dilatational wave ($\vartheta \neq 0$) causes a change in volume of the material elements in the body. Rotational wave (when $\vec{\psi} \neq 0$) produces a change in shape of the material element without changes in the volume of material elements.

Remark 2: Rotational waves are also referred as shear waves or a wave of distortion.

6.8 Plane Waves

A geometric surface of all points in space over which the phase of a wave is constant is called a wavefronts. Wavefronts can have many shapes. For example, wavefronts can be planes or spheres or cylinders. A line normal to the wave fronts, indicating the direction of motion of wave, is called a ray. If the waves are propagated in a single direction, the waves are called plane waves, and the wavefronts for plane waves are parallel planes with normal along the direction of propagation of the wave. Thus, a plane wave is a solution of the wave equation in which the disturbance/displacement varies only in the direction of wave propagation and is constant in all the directions orthogonal to propagation direction. The rays for plane waves are parallel straight lines.

6.9 Propagation of Plane elastic waves:

In the absence of body forces, vector equation of motion is

$$(\lambda+2\mu)\nabla(\nabla.\vec{u})-\mu\nabla\times(\nabla\times\vec{u})=\rho(\ddot{\vec{u}})$$

$$\Rightarrow \frac{(\lambda+2\mu)}{\rho}\nabla(\nabla.\vec{u})-\frac{\mu}{\rho}\nabla\times(\nabla\times\vec{u})=\ddot{\vec{u}}$$

$$\Rightarrow \alpha^2\nabla(\nabla.\vec{u})-\beta^2\nabla\times(\nabla\times\vec{u})=\ddot{\vec{u}} \qquad (1)$$

A solution of the equations of motion representing plane waves propagating in the direction

$\vec{p}=l_i\vec{e}_i$ with velocity c, is of the form

$$\vec{u} = \vec{u}\,\left(\vec{p}.\vec{R} - ct\right) \quad , \qquad \vec{R} = x_i \vec{e}_i$$
$$u_i = u_i\,\left(l_j x_j - ct\right)$$
$$\dot{u}_i = -c\,u_i'$$
$$\ddot{u}_i = c^2 u_i''$$
$$u_{i,j} = l_j u_i'$$
$$\nabla u_i = \vec{e}_j \frac{\partial u_i}{\partial x_j} = \vec{e}_j u_{i,j} = \vec{e}_j l_j u_i' = \vec{p} u_i'$$
$$\Rightarrow (\nabla u_i)\vec{e}_i = \vec{p}\,u_i'\,\vec{e}_i \quad \Rightarrow \quad \nabla\left(u_i \vec{e}_i\right) = \vec{p}\,u_i'\,\vec{e}_i$$

$$\Rightarrow \nabla \vec{u} = \vec{p}\vec{u}'$$

So gradient of a vector is dydic (2nd order tensor).

and

$$\nabla.\vec{u} = \vec{p}.\vec{u}' \quad , \qquad \nabla \times \vec{u} = \vec{p} \times \vec{u}'$$

$$\nabla \times \left(\nabla \times \vec{u}\right) = \vec{p} \times \left(\vec{p} \times \vec{u}''\right)$$
$$\nabla\left(\nabla.\vec{u}\right) = \vec{p}\left(\vec{p}.\vec{u}''\right)$$

Inserting these values in (1), we get

$$\alpha^2 \vec{p}\left(\vec{p}.\vec{u}''\right) - \beta^2 \vec{p} \times \left(\vec{p} \times \vec{u}''\right) = c^2 \vec{u}'' \qquad (2)$$

But $\vec{p} \times \left(\vec{p} \times \vec{u}''\right) = \vec{p}\left(\vec{p}.\vec{u}''\right) - \left(\vec{p}.\vec{p}\right)\vec{u}'' = \vec{p}\left(\vec{p}.\vec{u}''\right) - \vec{u}''$

Put this in (2), we get

$$\alpha^2 \vec{p}\,\left(\vec{p}.\vec{u}''\right) - \beta^2 \vec{p} \times \left(\vec{p} \times \vec{u}''\right) = c^2\left[\vec{p}\left(\vec{p}.\vec{u}''\right) - \vec{p} \times \left(\vec{p} \times \vec{u}''\right)\right]$$

or (3)

$$\left(c^2 - \alpha^2\right)\vec{p}\,\left(\vec{p}.\vec{u}''\right) - \left(c^2 - \beta^2\right)\vec{p} \times \left(\vec{p} \times \vec{u}''\right) = 0$$

$\Rightarrow$ either $c = \alpha$ and $\vec{p} \times \left(\vec{p} \times \vec{u}''\right) = 0$

$\Rightarrow c = \alpha$ and $\left(\vec{p} \times \vec{u}''\right)$ is arbitrary.

Or

$c = \beta \;, \quad (\vec{p}\,.\,\vec{u}'') = 0$

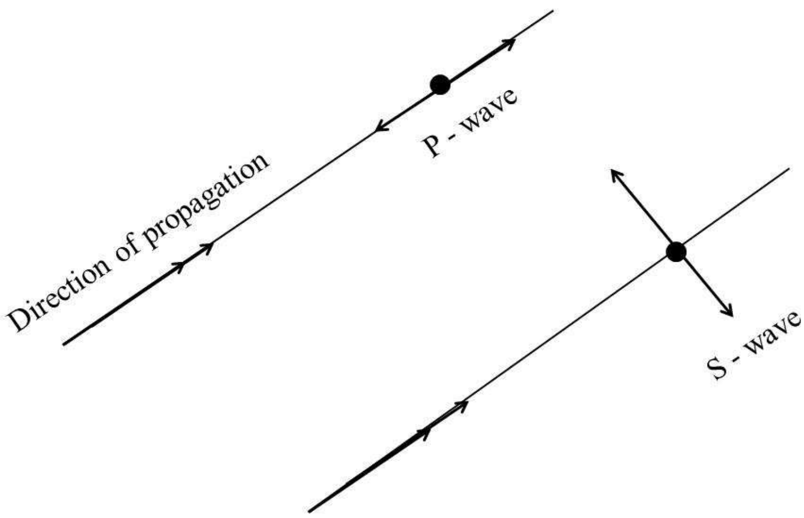
$\Rightarrow c = \alpha \;, \quad (\vec{p} \times \vec{u}) = 0$

$\Rightarrow \vec{p}$ and $\vec{u}$ are parallel.

and

$c = \beta \;, \quad (\vec{p}\,.\,\vec{u}) = 0$

$\Rightarrow \vec{p}$ and $\vec{u}$ are perpendicular (S-waves).

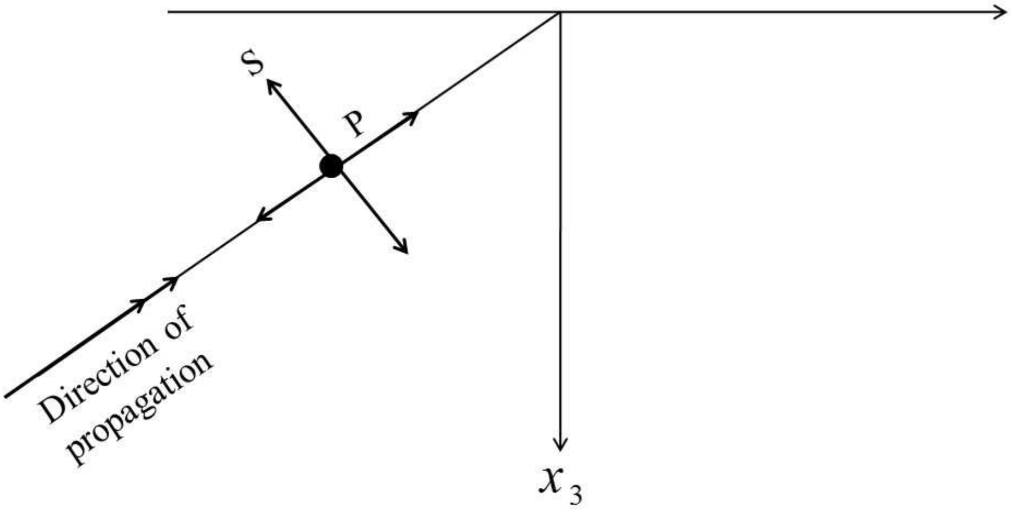


Therefore, for P-waves, the displacement vector $\vec{u}$ is parallel to the direction of propagation $\vec{p}$, i.e., P-waves are longitudinal waves.

Similarly for S-waves, the displacement vector $\vec{u}$ is perpendicular to the direction of propagation $\vec{p}$, i.e., S-waves are transverse waves.

6.10 P, SV and SH waves of Seismology:

Let vertical plane through the direction of propagation is x_1x_3-plane.



The displacement vector for S-waves can be decomposed into two components, one in the x_1x_3-plane known as SV-components and the other in the horizontal direction known as SV-components.

Here SV is vertically polarised components and SH is horizontal polarised components.

For P-waves, displacement is in the direction of propagation.

For SV-waves, displacement is perpendicular to the direction of propagation but in the vertical plane.

For SH-waves, displacement is in a horizontal direction perpendicular to the direction of propagation.

6.11 Wave propagation in 2-dimensions:

In 2-D motion, the motion is same in all planes parallel to a given plane. Let us take this plane as xz-plane.

Then $\frac{\partial}{\partial y} \equiv 0$

The equations of motion are

$$(\lambda + \mu)\vartheta_{,i} + \mu\nabla^2 u_i = \rho\ddot{u}_i \tag{1}$$

Let $u_i = (u, v, w)$

For i = 1, $(\lambda + \mu)\frac{\partial \vartheta}{\partial x} + \mu\nabla^2 u = \rho\ddot{u}$ (2)

For i = 2, $(\lambda + \mu)\frac{\partial \vartheta}{\partial y} + \mu\nabla^2 v = \rho\ddot{v} \;\Rightarrow\; \mu\nabla^2 v = \rho\ddot{v}$ (3)

For i = 3, $(\lambda + \mu)\frac{\partial \vartheta}{\partial z} + \mu\nabla^2 w = \rho\ddot{w}$ (4)

where

$$\vartheta = \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \tag{5}$$

and

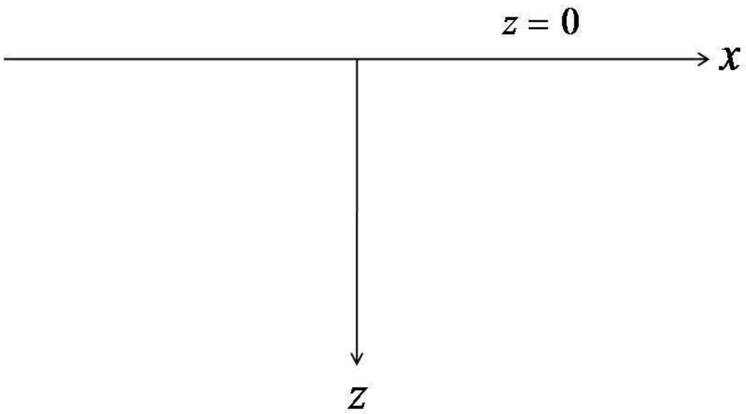
$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

Therefore, for 2-D motion parallel to xz-plane, equation of motion are (2), (3), (4).

v- Motion is known as Anti-plane motion or out of plane motion and (u, w)-motion is known as in plane motion.

6.12 Half-space Model or Semi-Infinite medium:

Boundary of medium is stress free, so $\tau_{zx}, \tau_{zy}, \tau_{zz}$ vanish.



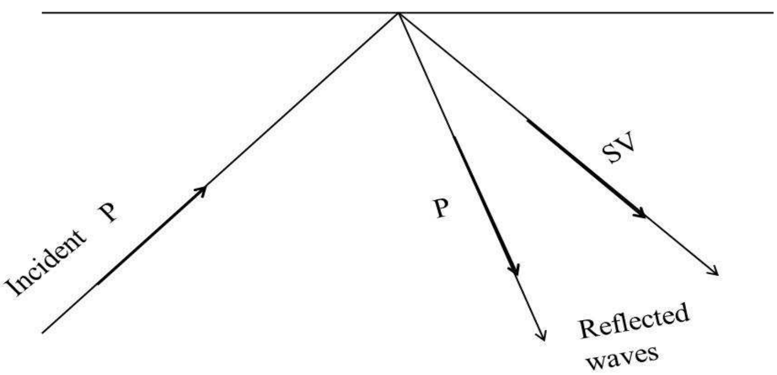
From the stress-strain relations, $\tau_{ij} = \lambda \vartheta \delta_{ij} + 2\mu e_{ij}$, we have

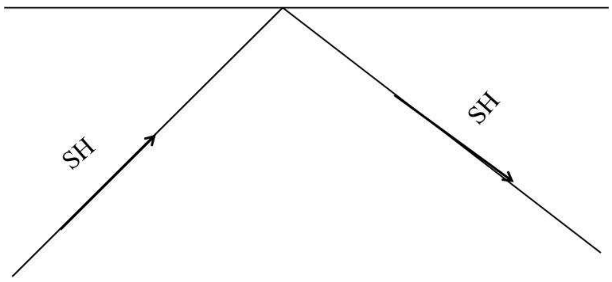
$$\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \tag{6}$$

$$\tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \equiv \mu \frac{\partial v}{\partial z} \tag{7}$$

$$\tau_{zz} = \lambda \frac{\partial u}{\partial x} + (\lambda + 2\mu)\frac{\partial w}{\partial z} \tag{8}$$

(P, SV) motion is independent of SH- motion, i.e., (u, w) motion is independent of v-motion.





6.13 Displacement potentials:

By Helmholtz's Theorem,

$$\vec{u} = \nabla\phi + curl\ \vec{\psi}\ , \quad div\,\vec{\psi} = 0 \tag{1}$$

The equations of motion in the absence of body forces are

$$(\lambda + \mu) grad\ div\vec{u} + \mu\nabla^2\vec{u} = \rho\ddot{\vec{u}}$$

or

$$(\lambda + 2\mu) grad\ div\,\vec{u} - \mu\, curl\, curl\,\vec{u} = \rho\ddot{\vec{u}} \tag{2}$$

$$\alpha^2\nabla(\nabla.\vec{u}) - \beta^2\nabla\times(\nabla\times\vec{u}) = \ddot{\vec{u}} \tag{3}$$

$$\alpha^2 = \frac{\lambda + 2\mu}{\rho}, \beta^2 = \frac{\mu}{\rho}$$

From (1) & (3), we get

$$\alpha^2\nabla(\nabla.(\nabla\phi + curl\,\vec{\psi})) - \beta^2\nabla\times(\nabla\times(\nabla\phi + curl\,\vec{\psi})) = \left(grad\,\frac{\partial^2\phi}{\partial t^2} + curl\,\frac{\partial^2\vec{\psi}}{\partial t^2} \right)$$

Since

$$div\, curl\,\vec{\psi} = 0,$$
$$curl\ grad\phi = 0$$

and

$$div(grad\phi) = \nabla.\nabla\phi = \nabla^2\phi$$

We have

$$\alpha^2\ grad\ \nabla^2\phi - \beta^2\ curl\ \left(-\nabla^2\vec{\psi} + grad\ div\vec{\psi}\right) = \left(grad\,\frac{\partial^2\phi}{\partial t^2} + curl\,\frac{\partial^2\vec{\psi}}{\partial t^2}\right)$$

$$\Rightarrow grad\ \left(\alpha^2\nabla^2\phi - \frac{\partial^2\phi}{\partial t^2}\right) + curl\ \left[\beta^2\nabla^2\vec{\psi} - \frac{\partial^2\vec{\psi}}{\partial t^2}\right] = 0$$

This equation is identically satisfied if ϕ and $\vec{\psi}$ satisfy the equations

$$\begin{aligned}\nabla^2\phi &= \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2}\\ \nabla^2\vec{\psi} &= \frac{1}{\beta^2}\frac{\partial^2\vec{\psi}}{\partial t^2}\end{aligned} \qquad (4)$$

Let (u, v, w) are components of displacements, then

From (1),

$$(u,v,w) = \vec{u} = \nabla\phi + curl\,\vec{\psi} \qquad [\vec{\psi} = (\psi_1,\psi_2,\psi_3)]$$

$$\Rightarrow u = \frac{\partial\phi}{\partial x} + \frac{\partial\psi_3}{\partial y} - \frac{\partial\psi_2}{\partial z}$$

$$v = \frac{\partial\phi}{\partial y} + \frac{\partial\psi_1}{\partial z} - \frac{\partial\psi_3}{\partial x}$$

$$w = \frac{\partial\phi}{\partial z} + \frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_1}{\partial y}$$

For (P-SV) motion,

$$u = \frac{\partial\phi}{\partial x} - \frac{\partial\psi_2}{\partial z}, \quad v = 0 \quad , \quad w = \frac{\partial\phi}{\partial z} + \frac{\partial\psi_2}{\partial x}$$

If $\psi_2 = -\psi$, then

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z} \quad , \quad v = 0 \; , \quad w = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x} \tag{5}$$

From (4),

$$\begin{aligned} \nabla^2 \phi &= \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2} \\ \text{and } \nabla^2 \psi &= \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2} \end{aligned} \tag{6}$$

Therefore, ϕ represents P-waves and ψ represents SV-waves,

The scalar potentials ϕ and ψ are known as displacement potentials.

The stresses are given by

$$\begin{aligned} \tau_{zx} &= \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\ \Rightarrow \tau_{zx} &= \mu \left[\frac{2\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \right] \end{aligned} \tag{7}$$

and

$$\tau_{zz} = \lambda \vartheta + 2\mu \frac{\partial w}{\partial z} = \lambda \nabla^2 \phi + 2\mu \left(\frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \psi}{\partial x \partial z} \right)$$

$$\Rightarrow \tau_{zz} = \mu \left[\frac{\lambda}{\mu} \nabla^2 \phi + 2 \left(\frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \psi}{\partial x \partial z} \right) \right]$$

but

$$\frac{\lambda}{\mu} = \frac{\lambda + 2\mu}{\mu} - 2 = \left(\frac{\alpha^2}{\beta^2} - 2 \right)$$

Therefore

$$\tau_{zz} = \mu \left[\left(\frac{\alpha^2}{\beta^2} - 2 \right) \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} \right) + 2 \left(\frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \psi}{\partial x \partial z} \right) \right]$$

$$= \mu \left[\left(\frac{\alpha^2}{\beta^2} - 2 \right) \frac{\partial^2 \phi}{\partial x^2} + \frac{\alpha^2}{\beta^2} \frac{\partial^2 \phi}{\partial z^2} - 2 \frac{\partial^2 \psi}{\partial x \partial z} \right]$$

and $\tau_{zy} = 0$

For SH-motion or V-motion:-

$$u = 0 \ , \quad w = 0$$

$$\nabla^2 v = \frac{1}{\beta^2} \frac{\partial^2 v}{\partial t^2}$$

$$\tau_{zy} = \mu \left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) = \mu \frac{\partial v}{\partial z}$$

$$\tau_{zx} = \tau_{zz} = 0$$

Note: Solution of one-dimensional wave equation:-

$$\phi = a \cos \left[mx - \omega t + \varepsilon \right]$$

$\Rightarrow \phi =$ Real part of $\left[A e^{i(\omega t - mx)} \right]$, where $A = a e^{-i\varepsilon}$, $a = |A|$; $\varepsilon = \arg A$

$$\Rightarrow \phi = \left[A e^{i(\omega t - mx)} \right]$$

i.e., ϕ satisfies one-dimensional wave equation, where m = constant, w = frequency and t = time.

6.14 Summary

We have studied about propagation of waves in an isotropic elastic solid medium and waves of dilatation and distortion. We also studied about elastic and plane elastic waves.

6.15 Keywords: Elastic wave, Plane wave, Wave propagation, P-waves, SH-wave, SV-wave

6.16 Self-assessment Questions

Q 1. What are plane waves? Derive the equation of plane waves.

Q 2. Define elastic waves and show that two types of elastic waves propagate in an infinite homogenous isotropic elastic medium.

Q 3. Describe wave motion in two dimension, in terms of displacement potential.

Q 4. Show that two types of waves can propagate in an unbounded homogenous isotropic elastic medium. Justify the nomenclature used to describe these waves.

6.17 Suggested Readings

1. I.S. Sokolnikoff, Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company Ltd., New Delhi.
2. Y.C. Fung, Foundations of Solid Mechanics, Prentice Hall, New Delhi.
3. S. Timoshenko and N. Goodier, Theory of Elasticity, McGraw Hill, New York.
4. Martin H. Sadd., Elasticity Theory, Applications and Numerics AP (Elsevier).
5. A.E.H. Love, A Treatise on the Mathematical Theory of Elasticity, 4^{th} Ed., Dover Publications, New York.

Chapter 7

Surface waves

7.1 Objectives

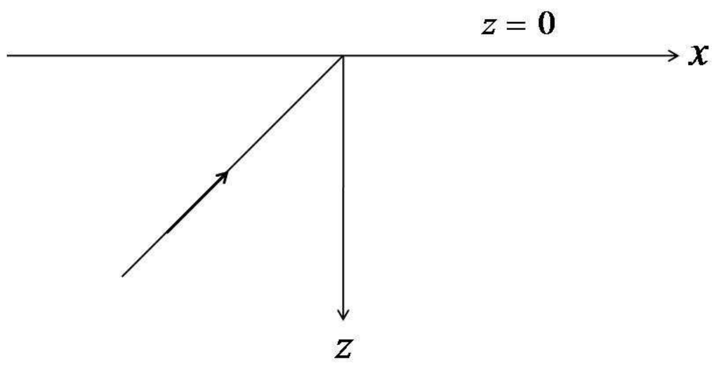
In this chapter, we shall discuss about surface waves, types of surface waves, Elastic surface waves such as Rayleigh and Love waves.

7.2 Introduction

In an elastic body, it is possible to have another type of waves (other than body waves) which are propagated over the surface and penetrate only a little into the interior of the body. Such waves are similar to waves produced on a smooth surface of water when a stone is thrown into it. These types of waves are called surface waves. Surface waves are "tied" to the surface and diminish exponentially as they get farther from the surface.

The criterion for surface waves is that the amplitude of the displacement in the medium dies exponentially with the increasing distance from the surface. In seismology, the interfaces are, in the ideal case, horizontal and so the plane of incidence is vertical. Activity of surface waves is restricted to the neighbourhood of the interface(s) or surface of the medium. Under certain conditions, such waves can propagate independently along the surface and interface. For surface waves, the disturbance is confined to a depth equal to a few wavelengths.

Let us take xz – plane as the plane of incidence with z – axis vertically downwards. Let z = 0 be the surface of a semi-infinite elastic medium (Figure).



For a surface wave, its amplitude will be a function of z (rather than an exponential function) which tends to zero as z→∞. For such surface waves, the motion will be two – dimensional, parallel to xz – plane, so that $\frac{\partial}{\partial y} = 0$.

7.3 Types of Surface waves

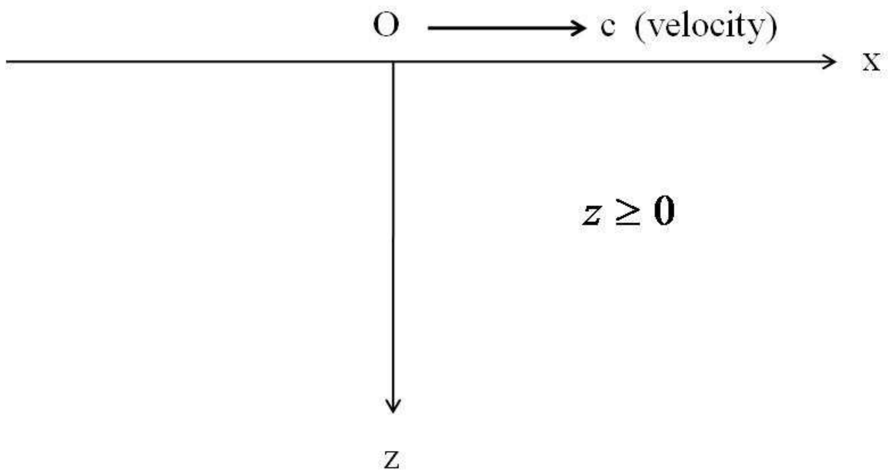
As the amplitude (disturbance) of Surface waves is significant only near the boundary and it decreases rapidly as we move away from the boundary. There are two types of surface waves:-

1. Surface waves of (P, SV) type are known as Rayleigh waves named after the scientist Rayleigh.
2. Surface waves of SH-type are known as LOVE waves named after AEH, Love.

7.4 Rayleigh waves

Rayleigh (1885) discussed the existence of a simplest surface wave propagating on the free – surface of a homogeneous isotropic elastic half – space.

We consider 2-D wave propagation in a homogeneous, isotropic and elastic half-space occupying the region z ≥ 0.



Then we have, displacement components are

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z}, \; v = 0, \quad w = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x}$$

Also ϕ and ψ satisfies the wave equations

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{1}{\alpha^2} \frac{\partial^2 \phi}{\partial t^2}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial t^2}$$

For Rayleigh wave propagating in the positive x- direction, we assume solution is of the form:

$$\phi(x, z, t) = f(z)\, e^{i\omega\left(t - \frac{x}{c}\right)}$$

or

$$\phi(x, z, t) = f(z)\, e^{i(\omega t - kx)}$$

For negative x-direction, we have solution is of the form:

$$\varphi(x,z,t) = f(z)e^{i\omega\left(t+\frac{x}{c}\right)}$$

where c is the velocity of propagation of Rayleigh waves and $k = \frac{\omega}{c}$ is wave number and ω is angular frequency.

Put the above in the following wave equation,

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{\alpha^2}\frac{\partial^2\phi}{\partial t^2}$$

We get

$$f(z)e^{i(\omega t-kx)}\left(-k^2\right) + e^{i(\omega t-kx)}\frac{d^2 f}{dz^2} = \frac{1}{\alpha^2}\left[f(z)e^{i(\omega t-kx)}\left(-\omega^2\right)\right]$$

$$\Rightarrow \quad -k^2 f + \frac{d^2 f}{dz^2} = \frac{1}{\alpha^2}\left(-\omega^2 f\right)$$

$$\Rightarrow \quad \frac{d^2 f}{dz^2} - \left(k^2 - \frac{\omega^2}{\alpha^2}\right)f = 0$$

$$\Rightarrow \quad \frac{d^2 f}{dz^2} - k^2\left(1 - \frac{\omega^2}{\alpha^2 k^2}\right)f = 0$$

$$\Rightarrow \quad \frac{d^2 f}{dz^2} - k^2 a^2 f = 0$$

where

$$a^2 = 1 - \frac{\omega^2}{\alpha^2 k^2} \qquad \text{or} \qquad a^2 = 1 - \frac{c^2}{\alpha^2}$$

$$\Rightarrow f(z) = e^{\pm kaz}$$

Therefore

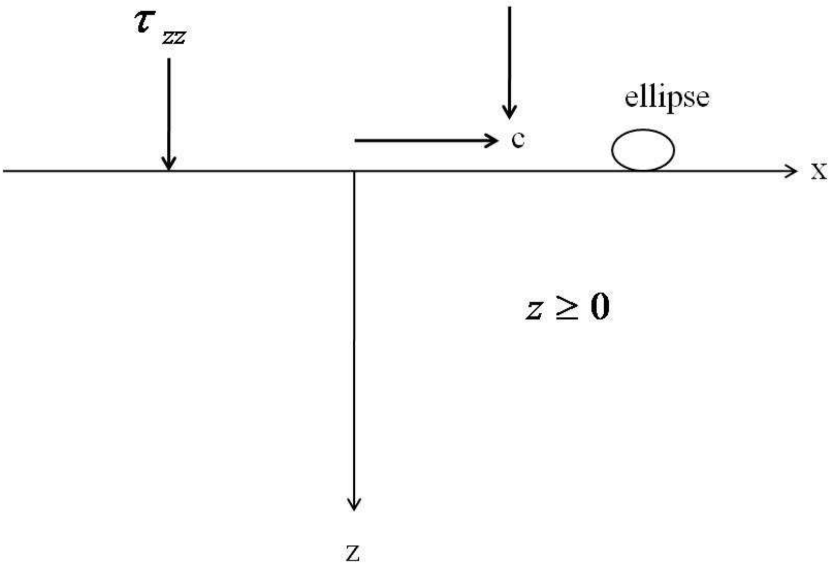
$\phi(x,z,t) = e^{i(\omega t - kx)} e^{\pm kaz}$

Similarly (replacing α by β),

$\psi(x,z,t) = e^{i(\omega t - kx)} e^{\pm kbz}$

where $b^2 = 1 - \dfrac{c^2}{\beta^2}$

To satisfy B.C., we neglect the positive sign.



Therefore, for Rayleigh waves propagating along the boundary of the half-space, z ≥ 0, we may assume

$$\phi = Ae^{i(\omega t - kx)} e^{-akz} \tag{1}$$

$$\psi = Be^{i(\omega t - kx)} e^{-bkz} \tag{2}$$

where

$$a = \sqrt{1 - \frac{c^2}{\alpha^2}} \quad , \quad b = \sqrt{1 - \frac{c^2}{\beta^2}}$$

a, b are real and A & B are arbitrary constants.

As a is real, $\Rightarrow c < \alpha$

and as b is real, $\Rightarrow c < \beta$

B.C., we assume that the surface of the half-space to be traction free, i.e., free boundary

$\Rightarrow \tau_{zz} = \tau_{zx} = 0$ at z = 0.

(here $\tau_{zy} = 0$, τ_{zy} is identically zero)

$$\tau_{zx} = \mu\left[\frac{2\partial^2\phi}{\partial x \partial z} - \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right]$$

and

$$\tau_{zz} = \mu\left[\left(\frac{\alpha^2}{\beta^2} - 2\right)\frac{\partial^2\phi}{\partial x^2} + \frac{\alpha^2}{\beta^2}\frac{\partial^2\phi}{\partial z^2} - 2\frac{\partial^2\psi}{\partial x \partial z}\right] \tag{3}$$

Ist B.C., $\tau_{zx} = 0$ at $z = 0$ (4)

$$\Rightarrow \left[\frac{2\partial^2\phi}{\partial x \partial z} - \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right] = 0 \text{ at } z = 0.$$

Using values of ϕ and ψ from (1) and (2), we get

$$2iaA + B\varsigma = 0 \tag{5}$$

where

$$\varsigma = 1 + b^2 = 2 - \frac{c^2}{\beta^2} \tag{6}$$

From (3), $\tau_{zz} = 0$ at $z = 0$, gives

$$\left[\left(\frac{\alpha^2}{\beta^2} - 2\right)\frac{\partial^2 \phi}{\partial x^2} + \frac{\alpha^2}{\beta^2}\frac{\partial^2 \phi}{\partial z^2} - 2\frac{\partial^2 \psi}{\partial x \partial z}\right] = 0$$

$$\Rightarrow \varsigma A - 2ibB = 0 \tag{7}$$

Equation (5) $\Rightarrow \varsigma = \dfrac{-2iaA}{B}$

Equation (7) $\Rightarrow \varsigma = \dfrac{2ibB}{A}$

Multiply these two equations, we get

$$\varsigma^2 = 4ab$$

Or $\left(2 - \dfrac{c^2}{\beta^2}\right)^2 = 4\sqrt{1 - \dfrac{c^2}{\alpha^2}}\sqrt{1 - \dfrac{c^2}{\beta^2}}$

On squaring, $\left(2 - \dfrac{c^2}{\beta^2}\right)^4 = 16\left(1 - \dfrac{c^2}{\alpha^2}\right)\left(1 - \dfrac{c^2}{\beta^2}\right)$ (8)

Equation (8) gives velocity of propagation of Rayleigh waves, since α, β are known, and so velocity for Rayleigh waves 'c' can be calculated.

This equation is known as Rayleigh wave equation. Equation (8) gives velocity of propagation of Rayleigh waves along the surface of a half-space.

We note that Equation (8) is independent of w, i.e. velocity c does not depend upon angular frequency w. Therefore, Rayleigh waves in a uniform half-space are Non-dispersive.

Solving (8) for c, we get

$$s^3 - 8s^2 + \left(24 - 16\frac{\beta^2}{\alpha^2}\right)s - 16\left(1 - \frac{\beta^2}{\alpha^2}\right) = 0, \qquad (9)$$

where $s = \dfrac{c^2}{\beta^2}$.

This is a cubic in s gives three solutions, either all real or one real and two complex.

Let $f(s) = s^3 - 8s^2 + \left(24 - 16\dfrac{\beta^2}{\alpha^2}\right)s - 16\left(1 - \dfrac{\beta^2}{\alpha^2}\right) = 0 \quad ; \quad (0 < s < 1)$

$f(0) = -16\left(1 - \dfrac{\beta^2}{\alpha^2}\right) < 0$ and $f(1) = 1 > 0$

Therefore

$f(s) = 0$ has either one or three roots satisfying the condition $0 < s < 1$.

$f''(s) = 6s - 16 = 0$ if $s = \dfrac{8}{3}$.

If the equation $f(s) = 0$ has three roots in (0, 1), then $f''(s) = 0$ must have one root in (0, 1).

Since $f''(s) = 0$ has no root in (0, 1). Therefore equation $f(s) = 0$ has one and only one root in (0, 1).

In case of Poissonian earth,

$\sigma = \frac{1}{4} = 0.25 \quad$ (since $\lambda = \mu$)

$\Rightarrow \frac{\alpha}{\beta} = \sqrt{3}$, $\frac{\beta^2}{\alpha^2} = \frac{1}{3}$

Put these values in equation (9), we get

s = 4, 3.15, 0.85 (10)

Only one root in (0, 1), i.e., s = 0.85 or $s = \frac{c^2}{\beta^2} = 0.85 \Rightarrow \frac{c}{\beta} = 0.92$ (11)

Therefore, in the Poisson's case, the velocity of propagation of Rayleigh waves is approximately equal to 0.92 times the velocity of propagation of S-waves.

We know that

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z} = -ikA\left(e^{-akz} - \frac{\varsigma}{2}e^{-bkz}\right) e^{i(wt-kx)} \quad \text{(using (6))} \qquad (12)$$

$$w = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x} = kA\left(-ae^{-akz} + \frac{2a}{\varsigma}e^{-bkz}\right) e^{i(wt-kx)} \quad \text{(using (5))} \qquad (13)$$

At the surface z = 0, we get

$$u_0 = -ikA\left(1 - \frac{\varsigma}{2}\right) e^{i\theta} \qquad \text{where } \theta = (wt - kx)$$

$$w_0 = akA\left(-1 + \frac{2}{\varsigma}\right) e^{i\theta} \qquad \text{where } \theta = (wt - kx)$$

Now $\left(1 - \frac{\varsigma}{2}\right) = \frac{c^2}{2\beta^2}$ and $\left(-1 + \frac{2}{\varsigma}\right) = \frac{\frac{c^2}{2\beta^2}}{1 - \frac{c^2}{2\beta^2}}$

Therefore,

$$u_0 = -ikAU(0)\, e^{i\theta} \quad , \quad w_0 = kAW(0)\, e^{i\theta} \tag{14}$$

where $U(0) = \dfrac{c^2}{2\beta^2}$, $W(0) = \dfrac{a\dfrac{c^2}{2\beta^2}}{1-\dfrac{c^2}{2\beta^2}} > 0 \qquad (\because c < \beta)$ (15)

Also we find that $U(0) < W(0)$

Taking the real part, (14) gives

$$u_0 = kAU(0)\sin\theta = b_1 \sin\theta,$$
$$w_0 = kAW(0)\cos\theta = a_1 \cos\theta \tag{16}$$

where
$b_1 = kAU(0)$, $a_1 = kAW(0) > b_1 \qquad (\because W(0) > U(0))$

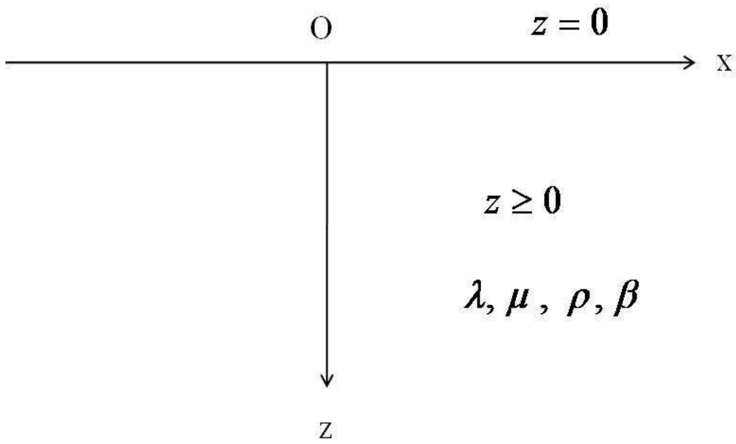
From (16),

$$\frac{u_0^2}{b_1^2} + \frac{w_0^2}{a_1^2} = 1$$

which is equation of an ellipse with a_1 and b_1 as semi-major and semi-minor axes. Therefore, surface particles describe ellipses. Here particle motion is elliptic retrograde (opposite to that of wave propagation).

7.5 Surface waves of SH-type or Love waves in a half-space model:

We consider first the possibility of the propagation of SH type surface waves (called Love waves) in a homogeneous semi-infinite isotropic elastic medium occupying the half-space z ≥ 0. The horizontal boundary z = 0 of the medium is assumed to be stress free. Let ρ be the density of the medium and λ, μ Lame's constants (as shown in figure).



Elastic isotropic half-space

Let the two – dimensional SH-wave motion takes place in the xz-plane. The basic equations for SH- wave motion are

$$u = 0 \ , \ w = 0 \ , \ v = v \ (x, z, t) \tag{1}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2} \tag{2}$$

$$\beta^2 = \frac{\mu}{\rho} \tag{3}$$

Let $v(x, z, t) = g(z)\, e^{i(wt - kx)}$

Then we get

$$-k^2 g + g'' = \frac{1}{\beta^2}\left(-w^2\right) g$$

$$\Rightarrow \frac{d^2 g}{dz^2} - k^2 \left(1 - \frac{c^2}{\beta^2}\right) g = 0$$

$$\Rightarrow \frac{d^2 g}{dz^2} - k^2 b^2 g = 0$$

$$\Rightarrow g(z) = e^{\pm kbz},$$

where

$$b^2 = 1 - \frac{c^2}{\beta^2}$$

Therefore

$$v = v(x, z, t) = e^{\pm kbz} e^{i(wt - kx)}$$

For possibility of existence of Love waves in a half-space model, we take

$$v = v(x, z, t) = e^{-kbz} e^{i(wt - kx)} \quad ; \quad c < \beta$$

B. C., $\tau_{zy} = 0$ at $z = 0$

$$\Rightarrow \mu \frac{\partial v}{\partial z} = 0 \quad \text{at} \quad z = 0$$

$$\Rightarrow \mu A\,(-kb)\, e^{i(wt - kx)} = 0 \Rightarrow A = 0$$

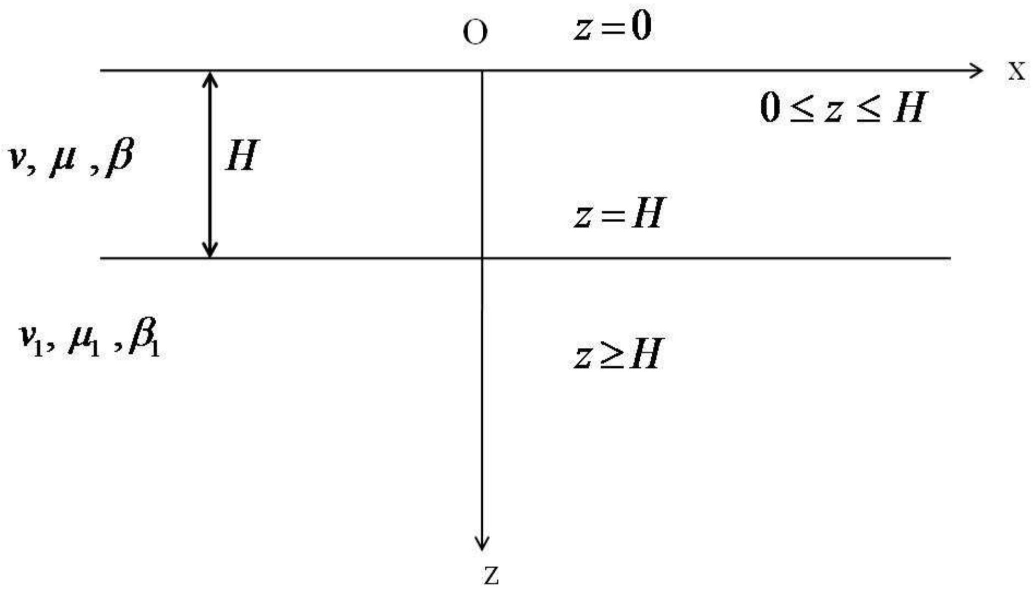
$\Longrightarrow$ Surface waves of SH-type or Love waves do not exist in a half-space model.

7.6 Propagation of Love Waves

These waves are named after AEH, Love (Surface wave of SH-type).

Surface waves of the SH-type are observed to occur on the earth's surface. Love (1911) showed that if the earth is modeled as an isotropic elastic layer of finite thickness lying over a homogeneous elastic isotropic half- space (rather than considering earth as a purely uniform half-space) then SH type waves occur in the stress-free surface of a layered half-space.

Let us consider a model consisting of a layer of uniform thickness H overlying a uniform half-space.



We assume that the layer and the half-space are in welded contact (displacement and stresses at this interface are continuous)

For layer $(0 \le z \le H)$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial z^2} = \frac{1}{\beta^2}\frac{\partial^2 v}{\partial t^2}$$

Let $v = \left(Ae^{-kbz} + Be^{kbz}\right) e^{i(wt-kx)}$; $c < \beta$ (1)

where $b = \sqrt{1 - \frac{c^2}{\beta^2}}$, $w = ck$

c is velocity of propagation of Love waves and

β is velocity of propagation of SH-waves in uniform layer of thickness H.

For half-space (z > H)

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial z^2} = \frac{1}{\beta_1^2} \frac{\partial^2 v_1}{\partial t^2}$$

Let $v_1 = A_1 e^{-kb_1 z} e^{i(wt-kx)}$; $c < \beta_1$ (2)

where $b_1 = \sqrt{1 - \frac{c^2}{\beta_1^2}}$, $c < \beta_1$

β_1 - Velocity of propagation of SH-waves in half-space.

B.C. (i) the surface z = 0 is traction free, i.e., $\tau_{zy} = 0$ at z = 0

$\left[\because \ \tau_{zx} \ \& \ \tau_{zz} \ \text{are identically zero}\right]$

$$\Rightarrow \mu \frac{\partial v}{\partial z} = 0 \qquad \text{at } z = 0$$

Using v from (1) at z = 0

$$\mu \left[A(-bk)e^{-bkz} + Bbke^{bkz}\right] e^{i(\omega t - kx)} = 0$$

$$\Rightarrow \mu \left[-Abk + Bbk\right] e^{i(\omega t - kx)} = 0 \quad \text{at} \quad z = 0$$

$$\Rightarrow -Abk + Bbk = 0$$

$$\Rightarrow A = B \qquad (3)$$

(ii) the displacement is continuous across the interface z = H,

i.e., $v = v_1$ at z = H

$$\Rightarrow Ae^{-bkH} + Be^{bkH} = A_1 e^{-kb_1 H} \qquad (4)$$

(iii) the traction is continuous across the interface z = H,

i.e., $\tau_{zy} = (\tau_{zy})_1$ at z = H

Here using for layer: τ_{zy} and for half-space: $(\tau_{zy})_1$

$$\Rightarrow \mu \frac{\partial v}{\partial z} = \mu_1 \frac{\partial v_1}{\partial z} \qquad \text{at } z = H$$

$$\Rightarrow \mu\left[-bAke^{-bkH} + kbBe^{bkH}\right] = k\mu_1 A_1 (-b_1) e^{-b_1 kH}$$

(5)

$$\Rightarrow Ae^{-bkH} - Be^{bkH} = \frac{\mu_1 b_1}{\mu b} A_1 e^{-b_1 kH}$$

Solving (3), (4), (5) for three unknowns, put B = A in (4) and (5) from (3), we get

$$A\left(e^{-bkH} + e^{bkH}\right) = A_1 e^{-b_1 kH}$$

and

$$A\left(e^{-bkH} - e^{bkH}\right) = \frac{\mu_1 b_1}{\mu b} A_1 e^{-b_1 kH}$$

On dividing, we get

$$\tan h(bkH) = -\frac{\mu_1 b_1}{\mu b}$$

$$\Rightarrow \tan h\left[kH\sqrt{1-\frac{c^2}{\beta^2}}\right] = -\frac{\mu_1}{\mu}\frac{\sqrt{1-\frac{c^2}{\beta_1^2}}}{\sqrt{1-\frac{c^2}{\beta^2}}} \qquad (6)$$

We assume that $c < \beta_1$ and consider following cases:

(i) If $c < \beta_1$, $c < \beta$

If $c < \beta$, then L.H.S. of (6) is real & positive and R.H.S. is real & negative.

Therefore, equation (6) has no real roots in c, i.e., c cannot be less than β.

(ii) If $c < \beta_1$, $c > \beta$

if $c > \beta$, $\sqrt{1-\frac{c^2}{\beta^2}} = i\sqrt{\frac{c^2}{\beta^2}-1}$

equation (6) becomes

$$\tan h\left[kHi\sqrt{\frac{c^2}{\beta^2}-1}\right] = i^2\frac{\mu_1}{\mu}\frac{\sqrt{1-c^2/\beta_1^2}}{i\sqrt{c^2/\beta^2-1}}$$

$$\Rightarrow i\tan\left[kH\sqrt{\frac{c^2}{\beta^2}-1}\right] = i\frac{\mu_1}{\mu}\frac{\sqrt{1-c^2/\beta_1^2}}{\sqrt{c^2/\beta^2-1}}$$

$$\Rightarrow \tan\left[kH\sqrt{\frac{c^2}{\beta^2}-1}\right] = \frac{\mu_1}{\mu}\frac{\sqrt{1-c^2/\beta_1^2}}{\sqrt{c^2/\beta^2-1}} \qquad (7)$$

This equation is known as Frequency equation or period equation or dispersion equation for Love waves in a layer of uniform thickness overlying a uniform half-space. Roots of this equation in c gives velocity of propagation.

So here $c < \beta_1$, $c > \beta$

$$\Rightarrow \beta < c < \beta_1$$

We note that for existence of Love waves, it is necessary that S-wave velocity in layer is less that the S-wave velocity in half-space. This gives the upper and lower bounds for the speed of Love waves.

From equation (7), we note that the velocity c depends on k or λ or ω (angular frequency), therefore, there is dispersion, i.e., Love waves are dispersive.

$$\left[\lambda = \frac{2\pi}{k}, \quad \omega = kc\right]$$

7.6 Summary

We have studied about surface waves, types of surface waves, Elastic surface waves such as Rayleigh and Love waves. The existence condition and nature of Rayleigh waves and Love waves have also been discussed.

7.7 Keywords: Surface waves, Rayleigh waves, Love waves.

7.8 Self-assessment Questions

Q 1. What are Surface waves? Derive the dispersion equation for Love waves in a layer of uniform thickness overlying a uniform half-space. Find the condition for the existence of real roots of this equation.

Q 2. Derive the equation giving the velocity of propagation of Love waves in a homogenous isotropic elastic layer over a homogenous isotropic half-space.

Q 3. Describe Surface waves. Explain the Rayleigh wave propagation in a homogenous elastic isotropic half-space, giving wave equation, and wave velocity and particle motion.

Q 4. Define P, SV and SH waves; Surface waves and plane waves.

7.9 Suggested Readings

1. I.S. Sokolnikoff, Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company Ltd., New Delhi.

2. Y.C. Fung, Foundations of Solid Mechanics, Prentice Hall, New Delhi.

3. S. Timoshenko and N. Goodier, Theory of Elasticity, McGraw Hill, New York.

4. Martin H. Sadd., Elasticity Theory, Applications and Numerics AP (Elsevier).

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Chapter-8

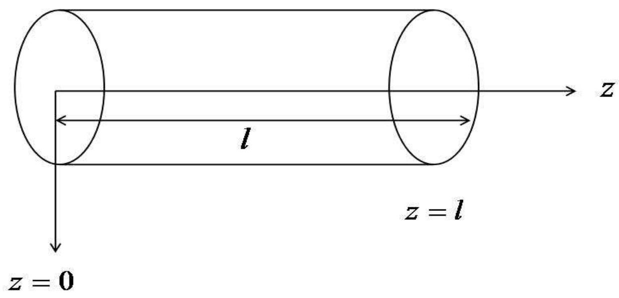
Torsion of Bars

8.1 Objectives

In this chapter, we shall discuss about torsion of cylindrical bars, Torsional rigidity, Torsion and stress functions, Lines of shearing stress. We will also study about simple problems related to circle, ellipse and equilateral triangle.

8.2 Introduction

Let us consider an elastic right circular beam of length *l*. We choose the z-axis along the axis of the beam so that its ends lie in the planes z = 0 and z = *l*, respectively. The end z = 0 is fixed in the xy-plane and a couple of vector moment $\vec{M} = M\,\hat{e}_3$ about the z-axis is applied at the end z = *l* . The lateral surface of the circular beam is stress-free and body forces are neglected.



The problem is to compute the displacements, strains and stresses developed in the beam because of the twist (or torsion) it experiences due to the applied couple.

(1) Equilibrium equations.

(2) Stress-strain (displacement) relation

(3) Boundary conditions

$T_i^{\nu} = \tau_{ij}\nu_j = 0$ on lateral surface (No external load or forces act on lateral surface)

On lateral surface, $\nu_3 = \nu_z = 0$

So boundary conditions are

$$\tau_{xx}\nu_x + \tau_{xy}\nu_y = 0$$
$$\tau_{yx}\nu_x + \tau_{yy}\nu_y = 0$$
$$\tau_{zx}\nu_x + \tau_{zy}\nu_y = 0$$

On lateral surface, these are satisfied.

(4) Compatibility equations

8.3 Torsion of cylindrical bars

Let us consider the torsion of non-circular cylinders.

Taking z-axis along the length of bar (beam) and one end of bar is fixed in the plane z = 0 (xy-plane) while other end in the plane z = l is twisted by a couple of magnitude M, whose moment is directed along the axis of the bar (i.e. z-axis). Thus, we assume that the displacement components are

$$u = -\alpha zy \quad , \quad v = \alpha zx \quad , \quad w = \alpha\,\phi(x,y) \qquad (1)$$

where $\phi(x,y)$ is some function of x & y and α is the twist per unit length of bar.

The function $\phi(x,y)$ must be determined as to satisfy the equilibrium equations, boundary equations and compatibility equations.

$$\tau_{ij} = \lambda u_{k,k}\delta_{ij} + \mu(u_{i,j} + u_{j,i})$$

$$\begin{aligned}
&\Rightarrow \tau_{yz} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = \mu\alpha\left(x + \frac{\partial \phi}{\partial y}\right) \\
&\tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = \mu\alpha\left(-y + \frac{\partial \phi}{\partial x}\right) \\
&\tau_{xx} = \tau_{yy} = \tau_{xy} = \tau_{zz} = 0
\end{aligned} \qquad (2)$$

If these stresses are used in equations of equilibrium in the absence of body forces,

$$\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{xy} + \frac{\partial}{\partial z}\tau_{xz} = 0 \qquad \text{(i)}$$

$$\frac{\partial}{\partial x}\tau_{yx} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{yz} = 0 \qquad \text{(ii)}$$

$$\frac{\partial}{\partial x}\tau_{zx} + \frac{\partial}{\partial y}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz} = 0 \qquad \text{(iii)}$$

We find that equations (i) & (ii) are satisfied and (iii) gives

$$\frac{\partial}{\partial x}\tau_{zx} + \frac{\partial}{\partial y}\tau_{yz} = 0$$

$$\Rightarrow \frac{\partial}{\partial x}\left[\mu\alpha\left(-y + \frac{\partial \phi}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\alpha\left(x + \frac{\partial \phi}{\partial y}\right)\right] = 0$$

$$\Rightarrow \mu\alpha\frac{\partial^2 \phi}{\partial x^2} + \mu\alpha\frac{\partial^2 \phi}{\partial y^2} = 0$$

So equations of equilibrium are satisfied if

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (3)$$

i.e., $\phi(x, y)$ satisfies 2-D Laplace equation

i.e., ϕ is harmonic function (Torsion function).

Boundary conditions are:

On lateral surface,

$$\tau_{xx}\nu_x + \tau_{xy}\nu_y = 0$$
$$\tau_{yx}\nu_x + \tau_{yy}\nu_y = 0$$
$$\tau_{zx}\nu_x + \tau_{zy}\nu_y = 0$$

1st and 2nd B.C. are satisfied while 3rd B.C. gives

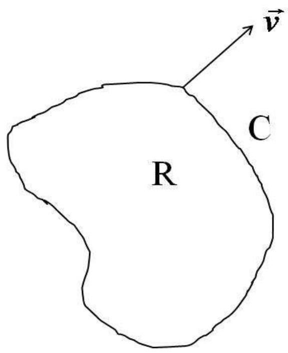
$$\left(-y + \frac{\partial\phi}{\partial x}\right)\nu_x + \left(x + \frac{\partial\phi}{\partial y}\right)\nu_y = 0$$

or

$$\frac{\partial\phi}{\partial x}\nu_x + \frac{\partial\phi}{\partial y}\nu_y = y\nu_x - x\nu_y$$

$$\Rightarrow \frac{d\phi}{d\nu} = y\nu_x - x\nu_y \qquad (4)$$

which is independent of z and equation (3) is also independent of z. So it becomes 2-D problem.



So

$$d\phi = y\nu_x - x\nu_y \qquad \text{on C}$$

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \qquad \text{in R}$$

Therefore, the problem of torsion of a beam (or bar) of arbitrary cross-section R bounded by C can be solved in terms of a function $\phi(x, y)$ such that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \qquad \text{in R}$$

and $d\phi = y\nu_x - x\nu_y$ on C.

This problem is called Neumann's problem.

If $\vec{F}$ is resultant force and $\vec{M}$ is resultant couple acting on base z = *l*, we have

$$F_x = \iint_R \tau_{zx}\, dx\, dy = \mu\alpha \iint_R \left(\frac{\partial\phi}{\partial x}-y\right) dx\, dy$$

$$= \mu\alpha \iint_R \left[\frac{\partial}{\partial x}\left\{x\left(\frac{\partial\phi}{\partial x}-y\right)\right\}+\frac{\partial}{\partial y}\left\{x\left(\frac{\partial\phi}{\partial y}+x\right)\right\}\right] dx\, dy$$

(By adding $x\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)$ in the integrand)

Apply Green's theorem,

$$\int_C (P dx + Q dy) = \iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right) dx\, dy,$$

We get

$$F_x = \mu\alpha \int_C \left[-x\left(\frac{\partial\phi}{\partial y} + x\right)dx + x\left(\frac{\partial\phi}{\partial x} - y\right)dy \right]$$

$$= \mu\alpha \int_C \left[-x\left(\frac{\partial\phi}{\partial y} + x\right)\frac{dx}{ds} + x\left(\frac{\partial\phi}{\partial x} - y\right)\frac{dy}{ds} \right] ds$$

Direction cosines of tangent are (x', y')

Direction cosines of normal are $(y', -x')$

Therefore

$$\nu_x = \frac{dy}{ds}, \qquad \nu_y = -\frac{dx}{ds}$$

Then using these, we get

$$F_x = 0$$

Similarly $F_y = 0$

$$F_z = \iint_R \tau_{zz}\, dx\, dy = 0$$

$\Rightarrow$ Resultant force is zero.

$$M_x = \iint_R \left(y\tau_{zz} - z\tau_{zy}\right) dx\, dy = -\iint_R z\tau_{zy}\, dx\, dy = -z\iint_R \tau_{zy}\, dx\, dy$$

$$\Rightarrow M_x = -z(0) = 0 \qquad \left[\because \quad \iint_R \tau_{zy}\, dx\, dy = F_y = 0\right]$$

Similarly,

$$M_y = 0$$

$$M_z = \iint_R \left(x\tau_{zy} - y\tau_{zx} \right) dx\, dy$$

$$= \mu\alpha \iint_R \left(x^2 + y^2 + x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x} \right) dx\, dy$$

Hence

$$M = \mu\alpha \iint_R \left(x^2 + y^2 + x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x} \right) dx\, dy$$

$$\Rightarrow M = D\alpha \qquad (5)$$

where $D = \mu \iint_R \left(x^2 + y^2 + x\frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial x} \right) dx\, dy$

and D is known as torsional rigidity of beam. It depends upon μ (rigidity) & shape of cross-section (region).

From (5), we have

$M \propto \alpha$

$\Longrightarrow$ The twisting moment M is proportional to the angle α of twist per unit length.

8.4 Stress function

Because torsion function is harmonic in R, we can construct an analytic function $(\phi + i\psi)$, where $\psi(x, y)$ is a conjugate harmonic function of $\phi(x, y)$.

i.e., $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0$ in R (6)

By C-R equations,

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \qquad \frac{\partial\phi}{\partial y} = \frac{-\partial\psi}{\partial x} \qquad (7)$$

But

$$\frac{d\phi}{d\nu} = \frac{\partial\phi}{\partial x}\nu_x + \frac{\partial\phi}{\partial y}\nu_y = \frac{\partial\psi}{\partial y}\nu_x - \frac{\partial\psi}{\partial x}\nu_y$$

$$= \frac{\partial\psi}{\partial y}\frac{dy}{ds} + \frac{\partial\psi}{\partial x}\frac{dx}{ds}$$

$$\Rightarrow \quad \frac{d\phi}{d\nu} = \frac{d\psi}{ds} \tag{8}$$

Equation (4) becomes

$$\frac{d\psi}{ds} = y\nu_x - x\nu_y \qquad \text{on} \quad C$$

$$\Rightarrow \quad \frac{d\psi}{ds} = y\frac{dy}{ds} + x\frac{dx}{ds} = \frac{d}{ds}\left[\frac{1}{2}\left(x^2 + y^2\right)\right]$$

On integrating, we get

$$\psi = \frac{1}{2}\left(x^2 + y^2\right) + \text{constant} \qquad \text{on} \quad C \tag{9}$$

Therefore, the torsion problem of a bar of arbitrary cross-section can be solved in terms of a function $\psi(x, y)$ s.t.

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad \text{in} \quad R$$

and $\psi = \frac{1}{2}\left(x^2 + y^2\right) + \text{constant} \qquad \text{on} \quad C$

Such type of problem is called Dirichlet's problem.

We introduce another function $\overline{\psi}$ (Introduced by Prandtl, L)

$$\overline{\psi} = \psi(x, y) - \frac{1}{2}\left(x^2 + y^2\right) \tag{10}$$

$$\Rightarrow \nabla^2\overline{\psi} = \frac{\partial^2\overline{\psi}}{\partial x^2} + \frac{\partial^2\overline{\psi}}{\partial y^2} = \nabla^2\psi - 2 \quad \text{in R}$$

But $\nabla^2\psi = 0$ in R

Then $\nabla^2\overline{\psi} = -2$ in R (11)

$\Rightarrow$ $\overline{\psi} =$ constant on C

Equation (11) is Poisson's equation.

Now

$$\tau_{yz} = \mu\alpha\left(x + \frac{\partial\phi}{\partial y}\right) = \mu\alpha\left(x - \frac{\partial\psi}{\partial x}\right) = -\mu\alpha\frac{\partial\overline{\psi}}{\partial x}$$

$$\tau_{xz} = \mu\alpha\left(-y + \frac{\partial\phi}{\partial x}\right) = \mu\alpha\left(-y + \frac{\partial\psi}{\partial y}\right) = \mu\alpha\frac{\partial\overline{\psi}}{\partial y} \qquad \text{[using (10)]}$$

As the stress components τ_{xz} & τ_{yz} are obtained from the function $\overline{\psi}(x, y)$, the function $\overline{\psi}(x, y)$ is called stress function.

If $\vec{T}$ is stress vector, then $\vec{T} = \hat{i}\,\tau_{zx} + \hat{j}\,\tau_{zy}$ is directed along the tangent to curve. Here normal stress is zero.

The curve $\overline{\psi}$ = constant are called lines of shearing stress.

If $\vec{T}$ is tangential stress and $\tau = \left|\vec{T}\right| = \sqrt{(\tau_{zx})^2 + (\tau_{zy})^2} = \mu\alpha\sqrt{\left(\frac{\partial\overline{\psi}}{\partial x}\right)^2 + \left(\frac{\partial\overline{\psi}}{\partial y}\right)^2}$

and maximum shearing stress occurs on the boundary C of the cross-section. To prove it, we shall use the following result:

1. Let a function $\Phi(x_1, x_2, x_3)$ is s.t.

(i) It is continuous and has continuous partial derivatives w.r.t. x_1, x_2, x_3 of first and 2nd order.

(ii) It is not identically equal to a constant.

(iii) It satisfies the inequality $\nabla^2\Phi \geq 0$ in R.

Then the function Φ attains its maximum value on the boundary C of the region R.

Here $\tau^2 = \mu^2\alpha^2\left[\overline{\psi}_x{}^2 + \overline{\psi}_y{}^2\right]$

$$\frac{\partial}{\partial x}\tau^2 = 2\mu^2\alpha^2\left[\overline{\psi}_x\overline{\psi}_{xx} + \overline{\psi}_y\overline{\psi}_{xy}\right]$$

$$\frac{\partial^2}{\partial x^2}\tau^2 = 2\mu^2\alpha^2\left[\overline{\psi}_{xx}^2 + \overline{\psi}_x\overline{\psi}_{xxx} + \overline{\psi}_y\overline{\psi}_{yxx} + \overline{\psi}_{yx}^2\right]$$

$$\frac{\partial^2}{\partial y^2}\tau^2 = 2\mu^2\alpha^2\left[\overline{\psi}_{yy}^2 + \overline{\psi}_x\overline{\psi}_{xyy} + \overline{\psi}_y\overline{\psi}_{yyy} + \overline{\psi}_{xy}^2\right]$$

Then $\nabla^2\tau^2 = 2\mu^2\alpha^2\left[\overline{\psi}_{xx}^2 + \overline{\psi}_x\left(\overline{\psi}_{xxx} + \overline{\psi}_{xyy}\right) + \overline{\psi}_y\left(\overline{\psi}_{yxx} + \overline{\psi}_{yyy}\right) + \overline{\psi}^2{}_{yy} + 2\overline{\psi}_{xy}^2\right]$

From equation (11), we have

$$\nabla^2\overline{\psi} = \overline{\psi}_{xx} + \overline{\psi}_{yy} = -2 \qquad \text{in R}$$

Differentiate w. r. t. x,

$$\left.\begin{aligned}\left(\overline{\psi}_{xxx} + \overline{\psi}_{xyy}\right) = 0\\ \left(\overline{\psi}_{yxx} + \overline{\psi}_{yyy}\right) = 0\end{aligned}\right\} \qquad \text{in R}$$

Then we get

$$\nabla^2 \tau^2 = 2\mu^2\alpha^2\left[\overline{\psi}_{xx}^2 + \overline{\psi}_{yy}^2 + 2\overline{\psi}_{xy}^2\right]$$

$$\Rightarrow \nabla^2 \tau^2 \geq 0$$

$\Longrightarrow$ max. shearing stress occurs on the boundary C of region R.

8.5 Torsion rigidity in terms of $\overline{\psi}$:-

Torsion rigidity is given by

$$M = D\alpha$$

or

$$D = \frac{1}{\alpha} M$$

[Here $M = M_z$]

$$\Rightarrow \quad D = \frac{1}{\alpha}\iint_R \left(x\tau_{zy} - y\tau_{zx}\right)dx\,dy$$

$$= -\mu \iint_R \left(x\frac{\partial\overline{\psi}}{\partial x} + y\frac{\partial\overline{\psi}}{\partial y}\right) dx\,dy$$

$$= -\mu \iint_R \left[\frac{\partial}{\partial x}\left(x\overline{\psi}\right) + \frac{\partial}{\partial y}\left(y\overline{\psi}\right)\right]dx\,dy + 2\mu\iint_R \overline{\psi}\, dx\,dy$$

Using Green's theorem,

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \int_C \left(Pdx + Qdy\right)$$

We get

$$D = -\mu\int_C \left(-y\overline{\psi}dx + x\overline{\psi}dy\right) + 2\mu\iint_R \overline{\psi}\, dx\,dy$$

$$= -\mu\int_C \overline{\psi}\left(-ydx + xdy\right) + 2\mu\iint_R \overline{\psi}\, dx\,dy$$

We can take $\overline{\psi} = 0$ on C.

Therefore

$$D = 2\mu \iint_R \overline{\psi} \; dx\, dy$$

So, the torsional rigidity D is twice the product of shear modulus μ & the volume enclosed by the surface $z = \overline{\psi}(x, y)$ and the plane z = 0.

8.6 Torsion of elliptic cylinder

Problem of torsion of a cylinder of any cross-section can be solved if we can find a function ψ such that

(i) $\nabla^2\psi$ or $\dfrac{\partial^2\psi}{\partial x^2} + \dfrac{\partial^2\psi}{\partial y^2} = 0$ in R (1)

and

(ii) $\psi(x, y) = \dfrac{1}{2}\left(x^2 + y^2\right)$ on C (2)

Let ϕ be conjugate harmonic function of ψ such that

$$\phi + i\,\psi = ic^2(x + iy)^2 + ik^2,$$

where c and k are constants.

This is the form of analytic function derived by Saint-Venant.

$$\phi + i\,\psi = ic^2\left(x^2 - y^2 + 2ixy\right) + ik^2$$

$$\Rightarrow \phi = -2c^2xy \tag{3}$$

$$\psi = c^2\left(x^2 - y^2\right) + k^2 \tag{4}$$

$$\frac{\partial \psi}{\partial x} = 2c^2 x, \quad \frac{\partial^2 \psi}{\partial x^2} = 2c^2$$

$$\frac{\partial \psi}{\partial y} = -2c^2 y, \quad \frac{\partial^2 \psi}{\partial y^2} = -2c^2$$

Therefore $\quad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 2c^2 - 2c^2 = 0$

From (2) & (4),

$$c^2\left(x^2 - y^2\right) + k^2 = \frac{1}{2}\left(x^2 + y^2\right) \qquad \text{on C}$$

$$\Rightarrow \left(\frac{1}{2} - c^2\right)x^2 + \left(\frac{1}{2} + c^2\right)y^2 = k^2$$

$$\Rightarrow \frac{x^2}{k^2 \Big/ \left(\frac{1}{2} - c^2\right)} + \frac{y^2}{k^2 \Big/ \left(\frac{1}{2} + c^2\right)} = 1 \qquad \text{on C} \tag{5}$$

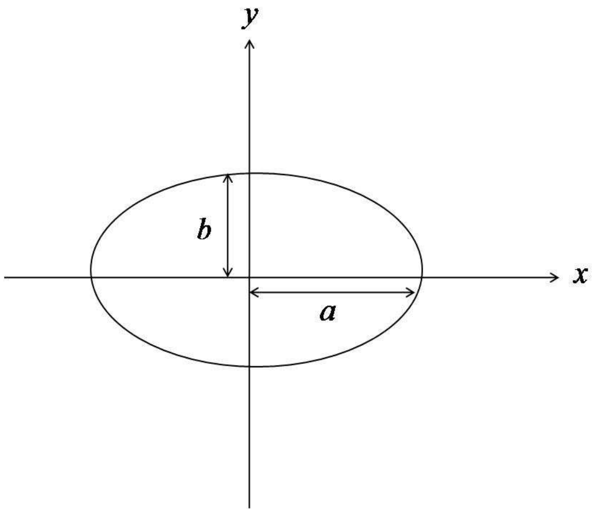
Let cross-section of cylinder (bar) be elliptic, then

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{6}$$

where

$$a^2 = \frac{k^2}{\frac{1}{2} - c^2} \tag{7}$$

and $\quad b^2 = \frac{k^2}{\frac{1}{2} + c^2} \qquad (8)$



For an ellipse, both *a* and *b* are positive. Then from (7), we have

$$c^2 < \frac{1}{2}$$

Or $\quad c < \frac{1}{\sqrt{2}}$

$$\frac{a^2}{b^2} = \frac{\frac{1}{2}+c^2}{\frac{1}{2}-c^2}$$

$$\Rightarrow c^2 = \frac{1}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)\ ,\ k^2 = \left(\frac{a^2b^2}{a^2+b^2}\right) \qquad (9)$$

Then (3) and (4) give

$$\phi = -\frac{\left(a^2-b^2\right)}{a^2+b^2}xy \qquad (10)$$

$$\psi = \frac{1}{2}\left(\frac{a^2-b^2}{a^2+b^2}\right)\left(x^2-y^2\right)+\frac{a^2b^2}{a^2+b^2} \qquad (11)$$

$$\tau_{yz} = \mu\alpha\left(x + \frac{\partial\phi}{\partial y}\right) = \mu\alpha\left[x - \frac{\left(a^2 - b^2\right)}{a^2 + b^2}x\right]$$

$$= \mu\alpha\, x\left[1 - \frac{\left(a^2 - b^2\right)}{a^2 + b^2}\right] = \mu\alpha\, x\frac{2b^2}{a^2 + b^2}$$

$$\Rightarrow \tau_{yz} = \frac{2\mu\alpha b^2 x}{a^2 + b^2} \qquad (12)$$

$$\tau_{xz} = \mu\alpha\left(-y + \frac{\partial\phi}{\partial x}\right) = \mu\alpha\left[-y - \frac{\left(a^2 - b^2\right)}{a^2 + b^2}y\right]$$

$$= \mu\alpha(-y)\left[1 + \frac{\left(a^2 - b^2\right)}{a^2 + b^2}\right] = \mu\alpha(-y)\frac{2a^2}{a^2 + b^2}$$

$$\Rightarrow \tau_{xz} = \frac{-2\mu\alpha a^2 y}{a^2 + b^2} \qquad (13)$$

Torsional moment $M = M_z = \iint_R \left(x\tau_{zy} - y\tau_{zx}\right)dx\,dy$

So $\quad M = M_z$

$$= \iint_R \left[x\left(\frac{2\mu\alpha b^2 x}{a^2 + b^2}\right) + y\frac{2\mu\alpha a^2 y}{a^2 + b^2}\right]dx\,dy$$

$$= \frac{2\mu\alpha}{a^2 + b^2}\iint_R \left(b^2 x^2 + a^2 y^2\right)\,dx\,dy$$

$$= \frac{2\mu\alpha}{a^2 + b^2}\left(b^2 \iint_R x^2 dx\,dy + a^2 \iint_R y^2\,dx\,dy\right)$$

Now

$$\iint_R x^2 \; dx\,dy \;=\; \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 \; dy\,dx$$

$$= 4 \int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} x^2 \; dy\,dx$$

$$= 4 \int_0^a x^2 \frac{b}{a}\sqrt{a^2-x^2} \; dx$$

$$= \frac{4b}{a} \int_0^a x^2 \sqrt{a^2-x^2} \; dx$$

Put $x = a\sin\theta \Rightarrow dx = a\cos\theta \; d\theta$

Therefore

$$\iint_R x^2 dx\,dy = \frac{4b}{a} \int_0^{\pi/2} a^2 \sin^2\theta\, a \cos\theta\, a \cos\theta \; d\theta$$

$$= 4ba^3 \int_0^{\pi/2} \sin^2\theta \cos^2\theta \; d\theta$$

$$= 4a^3 b \, \frac{1}{2} \; \frac{\Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}$$

$$= 2a^3 b \; \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right) \; \frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{2!}$$

$$= \frac{1}{4} a^3 b \pi = \frac{\pi a^3 b}{4}$$

Similarly $\iint_R y^2 dx\,dy = \frac{\pi a b^3}{4}$

Using these, we get

$$M_z = \frac{2\mu\alpha}{a^2+b^2}\left[b^2\frac{\pi a^3 b}{4} + a^2\frac{\pi a b^3}{4}\right]$$

$$= \frac{2\mu\alpha}{4(a^2+b^2)}\pi\left(a^3b^3 + a^3b^3\right) = \frac{4\mu\alpha\pi\, a^3b^3}{4(a^2+b^2)}$$

$$\Rightarrow \quad M_z = \frac{\mu\alpha\pi\, a^3b^3}{(a^2+b^2)}$$

Torsional moment $M = M_z = \iint_R \left(x\tau_{yz} - y\tau_{zx}\right) dx\ dy$

Therefore

$$M = \frac{2\mu\alpha b^2}{a^2+b^2}\iint_R x^2 dx\, dy + \frac{2\mu\alpha a^2}{a^2+b^2}\iint_R y^2 dx\, dy$$

$$= \frac{2\mu\alpha b^2}{a^2+b^2} I_y + \frac{2\mu\alpha a^2}{a^2+b^2} I_x$$

$$= \frac{2\mu\alpha}{a^2+b^2}\left[a^2 I_x + b^2 I_y\right]$$

where I_x and I_y are M.I. of elliptic section about x and y-axes, respectively.

We know that

$$I_x = \frac{\pi\, ab^3}{4}, \quad I_y = \frac{\pi\, ba^3}{4}$$

Therefore,

$$M = \frac{\pi\mu\alpha\, a^3b^3}{a^2+b^2} \tag{14}$$

Also, torsional rigidity D is given by

$$D = \frac{1}{\alpha} M$$

$$\Rightarrow D = \frac{\pi\mu\, a^3 b^3}{a^2 + b^2}$$

Stress function:-

$$\overline{\psi} = \psi - \frac{1}{2}\left(x^2 + y^2\right) = \frac{1}{2}\left(\frac{a^2 - b^2}{a^2 + b^2}\right)\left(x^2 - y^2\right) + \frac{a^2 b^2}{a^2 + b^2} - \frac{1}{2}\left(x^2 + y^2\right)$$

$$\Rightarrow \overline{\psi} = \frac{1}{2}\left(\frac{a^2 - b^2}{a^2 + b^2} - 1\right) x^2 - \frac{1}{2}\left(\frac{a^2 - b^2}{a^2 + b^2} + 1\right) y^2 + \frac{a^2 b^2}{a^2 + b^2}$$

$$= \frac{1}{2}\left(\frac{-2b^2}{a^2 + b^2}\right) x^2 - \frac{1}{2}\left(\frac{2a^2}{a^2 + b^2}\right) y^2 + \frac{a^2 b^2}{a^2 + b^2}$$

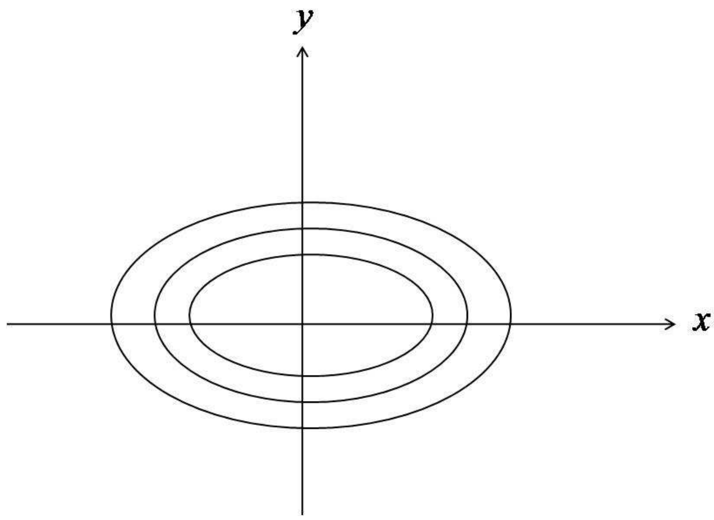
$$= \frac{-a^2 b^2}{a^2 + b^2}\left(\frac{x^2}{a^2}\right) - \frac{a^2 b^2}{a^2 + b^2}\left(\frac{y^2}{b^2}\right) + \frac{a^2 b^2}{a^2 + b^2}$$

$$\Rightarrow \overline{\psi} = \frac{-a^2 b^2}{a^2 + b^2}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) + \frac{a^2 b^2}{a^2 + b^2}$$

$\overline{\psi} =$ Constant = Lines of shearing stress

$$\Rightarrow \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \text{Constant}$$

$\Rightarrow$ Lines of shearing stress are family of confocal ellipses similarly to given ellipse.



Example: - Show that in the torsion of an elliptic cylinder,

$$\tau = \frac{2\mu\alpha ab}{a^2 + b^2}\sqrt{a^2 - e^2x^2}$$
$$e = \frac{1}{a}\sqrt{a^2 - b^2}$$

and max. shearing stress occurs on the end points of minor axes.

Solution: - We know that

$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} \qquad \text{[Using (12), (13)]}$$

$$\Rightarrow \quad \tau = \sqrt{\left(\frac{-2\mu\alpha a^2 y}{a^2 + b^2}\right)^2 + \left(\frac{2\mu\alpha b^2 x}{a^2 + b^2}\right)^2}$$
$$= \frac{2\mu\alpha}{a^2 + b^2}\sqrt{a^4y^2 + b^4x^2}$$

Since

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$\tau_{\text{bounbary}} = \frac{2\mu\alpha}{a^2+b^2}\sqrt{a^4b^2\left(1-\frac{x^2}{a^2}\right)+b^4x^2}$$

$$= \frac{2\mu\alpha ab}{a^2+b^2}\sqrt{a^2\left(1-\frac{x^2}{a^2}\right)+\frac{b^2}{a^2}x^2}$$

$$= \frac{2\mu\alpha ab}{a^2+b^2}\sqrt{a^2-x^2+\frac{b^2}{a^2}x^2}$$

$$= \frac{2\mu\alpha ab}{a^2+b^2}\sqrt{a^2-e^2x^2}$$

where e is eccentricity of ellipse and is given by

$$e^2 = 1-\frac{b^2}{a^2} = \frac{a^2-b^2}{a^2}$$

or

$$a^2e^2 = a^2-b^2 \qquad \Rightarrow \qquad b^2 = a^2\left(1-e^2\right)$$

Max. shearing stress occurs on boundary, $\tau_{\max.}$ when x is minimum.

So $\quad \tau_{\max} = \dfrac{2\mu\alpha a^2 b}{a^2+b^2}$

$\Rightarrow$ max. shearing stress occurs on the end points of minor axis.

8.7 Torsion of a triangular prism:

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = 0 \quad \text{in} \quad R \tag{1}$$

$$\psi = \frac{1}{2}\left(x^2+y^2\right) \quad \text{on} \quad C \tag{2}$$

For the equilateral triangular cross-section, we consider solution of $\nabla^2\psi = 0$ given

by

$$\phi + i\psi = ic(x+iy)^3 + ik \quad (3)$$

$$\Rightarrow \quad \phi + i\psi = ic\left(x^3 + 3ix^2y - 3xy^2 - iy^3\right) + ik$$

$$\Rightarrow \quad \phi = -3cx^2y + cy^3 \quad (4)$$

and $\psi = c\left(x^3 - 3xy^2\right) + k$

From (2) & (4), we get

$$c\left(x^3 - 3xy^2\right) + k = \frac{1}{2}\left(x^2 + y^2\right) \quad \text{on} \quad C \quad (5)$$

The line x = a will be part of boundary C if

$$c\left(a^3 - 3ay^2\right) + k = \frac{1}{2}\left(a^2 + y^2\right) \quad \forall y$$

$$\Rightarrow \quad -3ac = \frac{1}{2} \quad \text{and} \quad ca^3 + k = \frac{1}{2}a^2$$

$$\Rightarrow \quad c = \frac{-1}{6a} \quad \text{and} \quad k = \frac{2}{3}a^2 \quad (6)$$

Put values of c and k from (6) in (5), we get

$$\frac{-1}{6a}\left(x^3 - 3xy^2\right) + \frac{2}{3}a^2 = \frac{1}{2}\left(x^2 + y^2\right) \quad \text{on} \quad C$$

$$\Rightarrow x^3 + 3ax^2 - 3xy^2 + 3ay^2 - 4a^3 = 0 \quad \text{on} \quad C$$

This is cubic in x, but we know x = a is a part of boundary, so (x - a) is a factor.

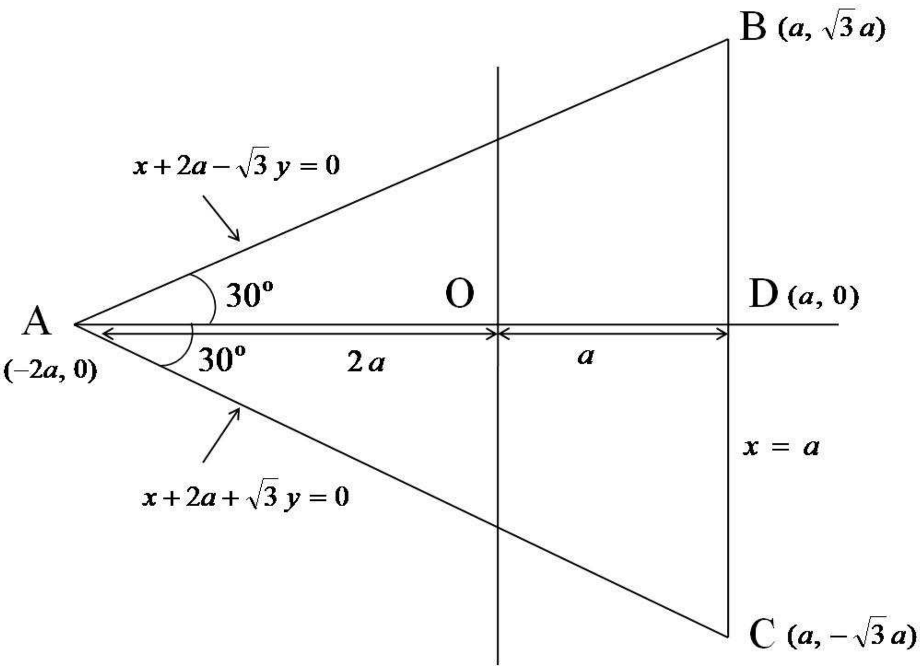
$$\Rightarrow \quad x^2(x-a) + 4ax(x-a) + 4a^2(x-a) - 3y^2(x-a) = 0$$

$$\Rightarrow \quad (x-a)\left[\left(x + 2a - \sqrt{3}y\right)\left(x + 2a + \sqrt{3}y\right)\right] = 0 \quad (7)$$

Therefore, boundary C consists of three straight lines

$$x - a = 0$$
$$x + 2a + \sqrt{3}y = 0 \tag{8}$$
$$x + 2a - \sqrt{3}y = 0$$

AD = 3a



$$\tau_{zx} = \mu\alpha\left(\frac{\partial\psi}{\partial y} - y\right)$$

where

$$\psi = \frac{-1}{6a}\left(x^3 - 3xy^2\right) + \frac{2}{3}a^2 \tag{9}$$

and $\phi = \frac{1}{2a}x^2 y - \frac{1}{6a}y^3$

Then

$$\tau_{zx} = \mu\alpha\left[\frac{-1}{6a}(-6xy) - y\right]$$
$$= \mu\alpha\left(\frac{xy}{a} - y\right)$$

$$\Rightarrow \qquad \tau_{zx} = \frac{\mu\alpha}{a}(xy - ay) = \frac{\mu\alpha}{a}[y(x-a)] \qquad (10)\ (i)$$

$$\tau_{zy} = \mu\alpha\left(-\frac{\partial\psi}{\partial x} + x\right)$$
$$= \mu\alpha\left[-\left\{\frac{-1}{6a}(3x^2 - 3y^2)\right\} + x\right]$$
$$= \mu\alpha\left[\frac{1}{2a}(x^2 - y^2) + x\right]$$

$$\Rightarrow \qquad \tau_{zy} = \frac{\mu\alpha}{2a}\left[x^2 + 2ax - y^2\right] \qquad (10)\ (ii)$$

Equation (10) gives tangential stresses at any line.

On the line x = *a*,

$$\tau_{zx} = 0$$
$$\tau_{zy} = \frac{\mu\alpha}{2a}(3a^2 - y^2)$$
$$\tau = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} = \frac{\mu\alpha}{2a}(3a^2 - y^2)$$

This is maximum when y is minimum (i.e., y = 0).

$$\tau_{\max} = \frac{3}{2}\mu\alpha a \qquad \text{(at } y = 0\text{)}$$

τ is zero at corner $(a, \sqrt{3}a)$

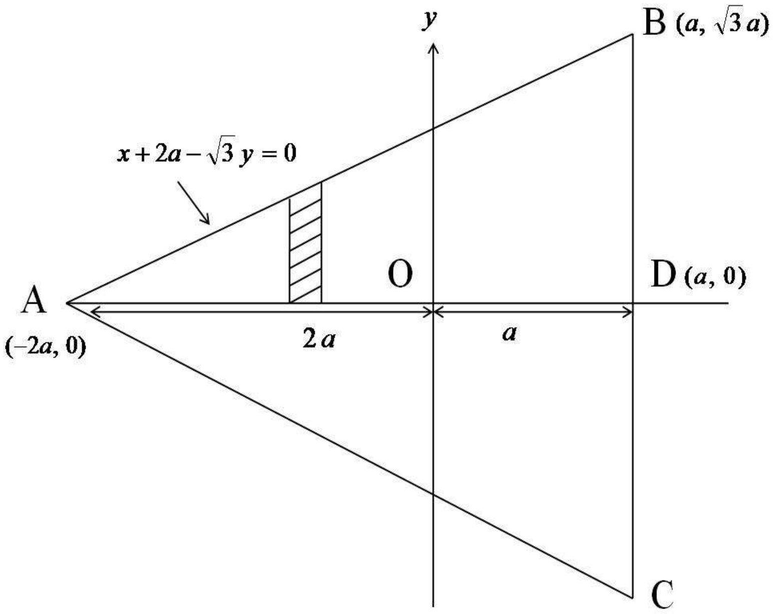
$\Rightarrow$ Stress is maximum at the midpoint of line x = a (for the line x - a = 0) , i.e.,

maximum at D and its value is $\frac{3}{2}\mu\alpha a$. Similarly, τ is maximum at other two sides

at mid-points (E and F) and value of maximum shearing stress is $\frac{3}{2}\mu\alpha a$.

Therefore, the shearing stress is maximum at the middle points of the sides of ΔABC

and maximum value is $\frac{3}{2}\mu\alpha a$.



$$M = M_z = \iint_R \left(x\tau_{zy} - y\tau_{zx}\right)dx\,dy = \frac{\mu\alpha}{2a}\iint_R \left(x^3 + 2ax^2 - 3xy^2 + 2ay^2\right)dx\,dy$$

Therefore

$$M=\frac{\mu\alpha(2)}{2a}\int_{x=-2a}^{a}\int_{y=0}^{\frac{x+2a}{\sqrt{3}}}\left(x^3+2ax^2-3xy^2+2ay^2\right)dydx$$

$$=\frac{\mu\alpha}{a}\int_{x=-2a}^{a}\left(x^3+2ax^2-\frac{3xy^2}{3}+\frac{2ay^2}{3}\right)[y]_0^{\frac{x+2a}{\sqrt{3}}}\,dx$$

$$=\frac{\mu\alpha}{a}\int_{-2a}^{a}\frac{(x+2a)}{\sqrt{3}}\left[x^3+2ax^2-\frac{3x(x+2a)^2}{3.3}+\frac{2a(x+2a)^2}{3.3}\right]dx$$

$$=\frac{\mu\alpha}{\sqrt{3}a}\int_{-2a}^{a}\left[x^3(x+2a)+2ax^2(x+2a)-\frac{x}{3}(x+2a)^3+\frac{2a}{3}(x+2a)^3\right]dx$$

$$=\frac{\mu\alpha}{\sqrt{3}a}\left[\frac{x^5}{5}+\frac{2ax^4}{4}+2a\left(\frac{x^4}{4}+\frac{2ax^3}{3}\right)-\frac{(x+2a)^4}{4.3}x+\frac{1}{60}(x+2a)^5+\frac{2a}{9}\frac{(x+2a)^4}{4}\right]_{-2a}^{a}$$

$$=\frac{\mu\alpha}{\sqrt{3}a}\left[\begin{matrix}\frac{a^5}{5}+\frac{1}{2}a^5+2a\left(\frac{a^4}{4}+\frac{2}{3}a^4\right)-\frac{(3a)^4a}{4.3}+\frac{a}{6.3}(3a)^4-\frac{(-2a)^5}{5}\\ -\frac{a}{2}(-2a)^4-2a\left\{\frac{(-2a)^4}{4}+\frac{2a}{3}(-2a)^3\right\}+\frac{1}{60}(3a)^5\end{matrix}\right]$$

$$=\frac{\mu\alpha}{\sqrt{3}a}\left[\frac{a^5}{5}+\frac{a^5}{2}+2a\frac{(3+8)a^4}{12}-\frac{81a^5}{12}+\frac{a}{18}81a^4+\frac{243}{60}a^5+\frac{32}{5}a^5-\frac{a}{2}\left(16a^4\right)-2a\left\{4a^4-\frac{16}{3}a^4\right\}\right]$$

On simplification, we get

$$M=\frac{9\sqrt{3}}{5}\mu\alpha\, a^4$$

Torsional rigidity, $D=\frac{1}{\alpha}M$

$$\Rightarrow\qquad D=\frac{9\sqrt{3}}{5}\mu a^4$$

Example: - Let D_0 denote the torsion rigidity of circular cylinder. Show that for

cross-sections of equal areas $D_e = kD_0$, $D_t = \frac{2\pi\sqrt{3}}{15} D_0$

where $K = \frac{2ab}{a^2 + b^2} \leq 1$

Solution:-

$D_0 = \frac{\pi}{2}\mu r^4$ for a circular cylinder of radius r.

$D_e = \frac{\pi\mu a^3 b^3}{a^2 + b^2}$ (Torsional rigidity of elliptic cylinder where a & b are semi-major & semi-minor axis of the ellipse, respectively.)

$D_t = \frac{9\sqrt{3}}{5}\mu x^4$ (Torsional rigidity of triangular lamina as Equilateral Δ of side $2\sqrt{3}$ x)

Since areas of cross-section are same, therefore

$$\underset{\text{(for circle)}}{\pi r^2} = \underset{\text{(for ellipse)}}{\pi ab} = \underset{\text{(for equilateral } \Delta)}{3\sqrt{3}x^2} \qquad (1)$$

$$\frac{D_e}{D_0} = \frac{\pi\mu a^3 b^3}{a^2 + b^2} \times \frac{2}{\pi\mu r^4} = \frac{2a^3 b^3}{(a^2 + b^2) r^4} = \frac{2a^3 b^3}{(a^2 + b^2) a^2 b^2} \qquad \text{[Using (1)]}$$

$$= \frac{2ab}{a^2 + b^2} = K$$

$$\Rightarrow \qquad D_e = KD_0$$

$$\frac{D_t}{D_0} = \frac{9\sqrt{3}}{5}\mu x^4 \times \frac{2}{\pi\mu r^4} = \frac{18\sqrt{3}}{5\pi} \times \frac{\pi^2}{9\times 3} \qquad \left[\text{Using (1) , } \frac{x^2}{r^2} = \frac{\pi}{3\sqrt{3}}\right]$$

$$\Rightarrow \qquad \frac{D_t}{D_0} = \frac{2\sqrt{3}}{15}$$

$$\Rightarrow \qquad D_t = \frac{2\sqrt{3}}{15} D_0$$

8.8 Summary

We have studied about torsion of cylindrical bars, Tortional rigidity, Torsion and stress functions, Lines of shearing stress. We have also studied about simple problems related to circle, ellipse and equilateral triangle.

8.9 Keywords: Torsion, Stress functions, cylindrical bars, Shearing stress, Ellipse.

8.10 Self-assessment Questions

Q 1. Derive the expression for torsional rigidity in case of the torsion of an elliptic cylinder.

Q 2. Derive the expression for torsional rigidity and twisting moment in case of the torsion of a cylindrical cylinder.

Q 3. Express torsional rigidity in terms of Stress function.

Q 4. Show that, in the torsion of an elliptic cylinder,

$$\tau = 2\mu\alpha \frac{ab}{a^2+b^2}\sqrt{a^2 - e^2x^2} \;; \qquad \text{where} \quad e = \frac{1}{a}\sqrt{a^2 - b^2}$$

and maximum shearing stress occurs on the end point of minor axes.

Q 5. Write a note on Prandtl stress Function.

Q 6. What is stress Function? Give its use.

8.11 Suggested Readings

1. I.S. Sokolnikoff, Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company Ltd., New Delhi.
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Chapter 9

Variational Methods

9.1 Objectives

In this chapter, we shall discuss the Variational Problems and Euler's Equations, Variational methods. We shall also discuss the minimum principles in deriving the equilibrium and compatibility equations of elasticity namely, Theorems of minimum potential energy, Theorems of minimum complementary energy, Reciprocal theorem of Betti and Rayleigh. Further we study some problems of Deflection of elastic string and elastic membrane by certain loads.

9.2 Introduction

The determination of the state of stress in the preceding chapters was made to depend on a solution of certain boundary- value problems involving partial differential equations. A different approach, exploiting certain broad minimum principles that characterize the equilibrium states of elastic bodies, is developed in this chapter.

We shall be using the minimum principles in deriving the equilibrium and compatibility equations of elasticity.

9.3 Variational Problems and Euler's Equations

We shall be concerned with the calculation of the extreme values of functions defined by certain integrals whose integrands contain one or several functions assuming the roles of arguments. As an example, consider the integral

$$I(y)=\int_{x_0}^{x_1} F(x,y,y')\,dx \qquad (1)$$

where $F(x,y,y')$ is a known real function F of the real arguments x, y and $y'=(dy/dx)$. The value of the integral (1) depends on the choice of $y = y(x)$, hence the notation I (y). We shall use the term *functional* to describe functions defined by integrals whose arguments themselves are functions.

For the meaningfulness of I (y), it is necessary to impose some restrictions on the choice of the argument $y(x)$, and on the prescribed function F appearing in the integrand of (1). It is assumed that at the end points of the interval (x_0, x_1), the specified values are y_0 and y_1.

Thus,

$$y(x_0) = y_0\ ,\quad y(x_1) = y_1 \qquad (2)$$

where y_0 and y_1 are prescribed values.

Further, for the integral (1) be minimized by the function $y = y(x)$, the necessary condition is

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad (3)$$

Equation (3) is the ***Euler's equation*** associated with the variational problem I (y) = minimum expressed by equation (1).

On expanding (3), we get the second order ordinary differential equation

$$y''\frac{\partial^2 F}{\partial y'^2}+y'\frac{\partial^2 F}{\partial y\,\partial y'}+\frac{\partial^2 F}{\partial x\,\partial y'}-\frac{\partial F}{\partial y}=\ 0 \tag{4}$$

for the determination of $y(x)$.

Similar calculations performed on the functional

$$I(y)=\int_{x_0}^{x_1} F(x,y,y',y'',..........,y^{(n)})\,dx \tag{5}$$

yield the Euler equation

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)-\,\ldots\ldots\ldots+(-1)^n\,\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^n}\right)=\ 0\quad , \tag{6}$$

when certain obvious restrictions on the continuity and differentiability of F and $y(x)$ are imposed.

We consider next the problem of minimizing the double integral

$$I(u)=\iint_R F(x,y,u,u_x\,,u_y)\ dx\ dy \tag{7}$$

on the set $\{u\,(x,y)\}$ of functions, where each $u\,(x,y)$ in the set takes on the boundary C of the region R specified continuous values $u=\phi(s)$.

The condition for the minimizing function $u\,(x,y)$ expressed by equation (7) is

$$\frac{\partial F}{\partial u}-\ \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right)-\ \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right)=\ 0 \tag{8}$$

Similarly for the double integral

$$I(u)=\iint_R F(x,y,u,u_x\,,u_y\,,u_{xx}\,,u_{xy}\,,u_{yy})\ dx\ dy\ , \tag{9}$$

the condition for the minimizing function $u(x, y)$ is

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial F}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial u_{xy}}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{\partial F}{\partial u_{yy}}\right) = 0 \qquad (10)$$

Note: Poisson's equation $\nabla^2 u = f$,

with given Boundary conditions, is an Euler's equation of variational problem

$$I[u(x, y)] = \iint_R \left[u_x^{\ 2} + u_y^{\ 2} + 2\, fu\right] dx\ dy = \min.$$

Solution: Given variational problem is

$$I[u(x, y)] = \iint_R \left[u_x^{\ 2} + u_y^{\ 2} + 2\, fu\right] dx\ dy = \min. \qquad (1)$$

Here $F = u_x^{\ 2} + u_y^{\ 2} + 2\, fu$ (2)

Then the Euler's equation

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

becomes

$$2\, f - \frac{\partial(2u_x)}{\partial x} - \frac{\partial(2u_y)}{\partial y} = 0$$

$$\Rightarrow\ u_{xx} + u_{yy} = f$$

which is Poisson's equation $\nabla^2 u = f$

9.4 Theorem of minimum potential energy

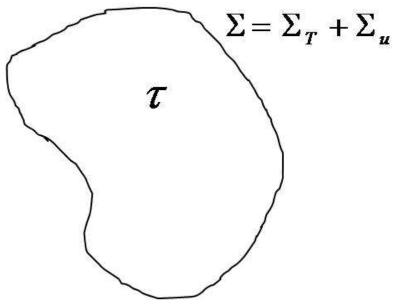
Now, we introduce an important functional, called the potential energy of deformation, and prove that this functional attains an absolute minimum when the displacements of the elastic body are those of the equilibrium configuration.

Statement: Of all displacements satisfying the given Boundary conditions, those which satisfy the equilibrium equations make the potential energy as absolute minimum.

Proof: If T_i are the surface forces

and F_i are the body forces

Also T_i are given over surface $\sum_T$ and displacement are prescribed over $\sum_u$.



The displacement for equilibrium state are u_i and arbitrary displacement $u_i + \delta u_i$, consistent with constraints imposed on the body, i.e., over the portion $\sum_u$ of $\sum$, where displacement are given, $\delta u_i = 0$ but over the part $\sum_T$, δu_i are arbitrary and we call these arbitrary displacement δu_i, the virtual displacement. The virtual work δU done by external forces F_i and T_i in a displacement δu_i is defined by equation:

$$\delta U = \int_{\Sigma} T_i \ \delta u_i \ d\Sigma + \int_{\tau} F_i \ \delta u_i \ d\tau \qquad (1)$$

Also strain energy U is given by

$$U = \int_{\tau} W \, d\tau \qquad (A)$$

where

$$W = \frac{\lambda}{2}\vartheta^2 + \mu e_{ij} e_{ij} \qquad (B)$$

Since τ is fixed, T_i and F_i do not vary when we consider the displacement δu_i, (1) can be written as

$$\delta U = \delta\left[\int_{\Sigma} T_i u_i \, d\Sigma \ + \int_{\tau} F_i u_i \ d\tau\right] \qquad (2)$$

From (A),

$$\delta U = \delta \int_{\tau} W \, d\tau \qquad (3)$$

From (2), (3), we get

$$\delta\left(\int_{\tau} W \, d\tau - \int_{\Sigma} T_i u_i \, d\Sigma \ - \int_{\tau} F_i u_i \ d\tau\right) = 0 \qquad (4)$$

$\Rightarrow \int_{\tau} W \ d\tau - \int_{\Sigma} T_i u_i \ d\Sigma \ - \int_{\tau} F_i u_i \ d\tau$ has a stationary value.

If we define the Potential Energy, V by

$$V = \int_{\tau} W \ d\tau - \int_{\Sigma} T_i u_i \ d\Sigma - \int_{\tau} F_i u_i \ d\tau \qquad (5)$$

Equation (4) gives, $\delta V = 0$ (6)

Next we prove that the functional V assumes a minimum value when the displacement u_i are those of equilibrium state.

To prove this, we will prove that increment ΔV produced in V by replacing equilibrium displacement u_i by $u_i + \delta u_i$ is positive for all non-vanishing δu_i.

$$\Delta W = \frac{\lambda}{2}\vartheta^2 + \mu e_{ij}e_{ij}\Big|_{u+\delta u} - \left(\frac{\lambda}{2}\vartheta^2 + \mu e_{ij}e_{ij}\right)\Big|_u$$

We know that $e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

$$e_{ij}\Big|_{u+\delta u} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) + \frac{1}{2}\left[\left(\delta u_i\right)_{,j} + \left(\delta u_j\right)_{,i}\right]$$

$$\Rightarrow e_{ij}\Big|_{u+\delta u} = e_{ij} + \frac{1}{2}\left(\delta u_i\right)_{,j} + \frac{1}{2}\left(\delta u_j\right)_{,i}$$

$$\vartheta\Big|_{u+\delta u} = e_{ii} + \left(\delta u_i\right)_{,i} = \vartheta + \left(\delta u_i\right)_{,i}$$

$$\Rightarrow \Delta W = \frac{\lambda}{2}\left[\vartheta + \left(\delta u_i\right)_{,i}\right]\left[\vartheta + \left(\delta u_i\right)_{,i}\right]$$

$$+ \mu\left[e_{ij} + \frac{1}{2}\left(\delta u_i\right)_{,j} + \frac{1}{2}\left(\delta u_j\right)_{,i}\right]\left[e_{ij} + \frac{1}{2}\left(\delta u_i\right)_{,j} + \frac{1}{2}\left(\delta u_j\right)_{,i}\right] - \frac{\lambda}{2}\vartheta^2 - \mu e_{ij}e_{ij}$$

$$\Rightarrow \Delta W = \lambda\vartheta\left(\delta u_i\right)_{,i} + 2\mu e_{ij}\left(\delta u_i\right)_{,j} + P \qquad (7)$$

where

$$P = \frac{\lambda}{2}\left[\left(\delta u_i\right)_{,i}\right]^2 + \frac{\mu}{4}\left[\left(\delta u_i\right)_{,j} + \left(\delta u_j\right)_{,i}\right]^2 \geq 0 \qquad (8)$$

Or

(7) $\Rightarrow$ $\qquad \Delta W = \left(\lambda\vartheta\delta_{ij} + 2\mu e_{ij}\right)\left(\delta u_i\right)_{,j} + P \qquad \left(\because \delta_{ij}\left(\delta u_i\right)_{,j} = \left(\delta u_i\right)_{,i}\right)$

$$\Rightarrow \qquad \Delta W = \tau_{ij}\left(\delta u_i\right)_{,j} + P$$

Also

$$\begin{aligned}
\Delta U &= \int_\tau \Delta W\, d\tau \\
&= \int_\tau \tau_{ij}\left(\delta u_i\right)_{,j} d\tau + \int_\tau P\, d\tau \\
&= \int_\tau \left(\tau_{ij}\delta u_i\right)_{,j} d\tau - \int_\tau \left(\tau_{ij,j}\delta u_i\right) d\tau + \int_\tau P\, d\tau \\
&= \int_\Sigma \left(\tau_{ij}\nu_j\delta u_i\right) d\Sigma - \int_\tau \left(\tau_{ij,j}\delta u_i\right) d\tau + \int_\tau P\, d\tau \qquad \text{(by Gauss's divergence theorem)}
\end{aligned}$$

If body is in equilibrium, then

$\tau_{ij,j} + F_i = 0$ in τ

$\Rightarrow \quad \tau_{ij,j} = -F_i$ in τ

and $\tau_{ij}\nu_j = T_i$ on Σ

Therefore

$$\Delta U = \int_\Sigma T_i\left(\delta u_i\right) d\Sigma + \int_\tau \left(F_i\delta u_i\right) d\tau + Q \quad ; \qquad Q \geq 0 \qquad (9)$$

But $\Delta V = \Delta U - \int_\Sigma T_i\left(\delta u_i\right) d\Sigma - \int_\tau \left(F_i\,\delta u_i\right) d\tau$

Put this in (9), we get

$\Delta V = Q$, since $Q \geq 0$ where $Q = \int_\tau P d\tau \geq 0$

Hence ΔV is positive. Hence the theorem.

9.5 Reciprocal theorem of Betti and Rayleigh or Betti's Reciprocal theorem or theorem of work and reciprocity

Statement : If an elastic body is subjected to two system of body and surface forces, then the work done by the first set of forces (T_i, F_i) acting over the displacements u_i' produced by the second set is equal to work done by the second set of forces (T_i', F_i') over the displacements u_i produced by the first set of forces, i.e., prove that

$$\int_\tau \left(F_i u_i'\right) d\tau + \int_\sigma \left(T_i u_i'\right) d\sigma = \int_\tau \left(F_i' u_i\right) d\tau + \int_\sigma \left(T_i' u_i\right) d\sigma \tag{1}$$

Proof: - Equilibrium equations for the two systems of forces are

$$\tau_{ij,j} + F_i = 0 \tag{2}$$

$$\tau'_{ij,j} + F_i' = 0 \tag{3}$$

L.H.S of (1),

$$\int_\tau \left(F_i u_i'\right) d\tau + \int_\sigma \left(T_i u_i'\right) d\sigma = \int_\tau \left(F_i u_i'\right) d\tau + \int_\sigma \left(\tau_{ij} \nu_j u_i'\right) d\sigma$$

$$= \int_\tau \left(F_i u_i'\right) d\tau + \int_\tau \left(\tau_{ij} u_i'\right)_{,j} d\tau \qquad \text{(By Gauss divergence theorem)}$$

$$= \int_\tau \left(F_i u_i'\right) d\tau + \int_\tau \left(\tau_{ij,j} u_i' + \tau_{ij} u_{i,j}'\right) d\tau$$

$$= \int_\tau \left[\left(F_i u_i' + \tau_{ij,j} u_i'\right) + \tau_{ij} e_{ij}'\right] d\tau \qquad \left[\because \begin{array}{l} \tau_{ij} u_{i,j}' = \tau_{ij} e_{ij}' + \tau_{ij} w_{ij}' = \tau_{ij} e_{ij}' \\ \text{and} \quad \tau_{ij,j} + F_i = 0 \end{array}\right]$$

Therefore

L.H.S. of (1) $= \int_\tau \tau_{ij} e_{ij}' \, d\tau$

$$= \int_\tau e_{ij}' \left(\lambda \vartheta \delta_{ij} + 2\mu e_{ij}\right) d\tau$$

$$= \int_\tau \left(\lambda \vartheta \vartheta' + 2\mu e_{ij} e_{ij}'\right) \, d\tau \qquad [\because \quad \delta_{ij} e_{ij}' = e_{ii}' = \vartheta']$$

Here Integrand is symmetric in prime and unprime variable.

Therefore

R.H.S. of (1) $= \int_\tau \left(\lambda \vartheta' \vartheta + 2\mu \, e_{ij}' e_{ij}\right) d\tau$.

Hence $\int_\tau \left(F_i \, u_i'\right) d\tau + \int_\sigma \left(T_i \, u_i'\right) d\sigma = \int_\tau \left(F_i' \, u_i\right) d\tau + \int_\sigma \left(T_i' \, u_i\right) \, d\sigma$.

9.6 Theorem of Minimum complementary energy

Definition: The complementary energy V* is defined by the formula

$$V^* = U - \int_{\Sigma_u} \left(u_i \, T_i\right) d\Sigma \; = \; \int_\tau W \; d\tau - \int_{\Sigma_u} \left(u_i \, T_i\right) d\Sigma$$

where U is the strain energy and W is the strain energy function.

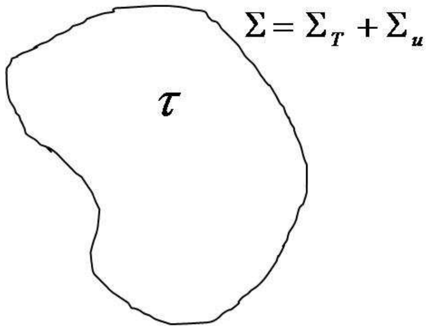
Statement: The complementary energy V^* has an absolute minimum when the stress tensor τ_{ij} is that of equilibrium state and the varied state of stress satisfy the following condition:

(i) $\left(\delta\tau_{ij}\right)_{,j} = 0$ in τ

(ii) $\left(\delta\tau_{ij}\right) \nu_j = 0$ on Σ_T

(iii) $\left(\delta\tau_{ij}\right)$ is arbitrary on Σ_u

where T_i are prescribed surface forces and F_i are body forces.



Proof: - Let stresses corresponding to equilibrium state are denoted by τ_{ij}, then equilibrium equations are satisfied. So

$$\tau_{ij,j} + F_i = 0 \quad \text{in} \quad \tau \tag{1}$$

$$\tau_{ij}\nu_j = T_i \quad \text{on} \quad \Sigma_T \tag{2}$$

$$u_i = f_i \quad \text{on} \quad \Sigma_u$$

For varied state, $\tau_{ij}' = \tau_{ij} + \delta\tau_{ij}$ (3)

so that

(i) $\tau'_{ij,j} + F_i = 0$ in τ (4)

(ii) $\tau'_{ij}\nu_j = T_i$ on Σ_T (5)

(iii) On the part Σ_u of Σ, τ_{ij}' are arbitrary.

$$(\delta\tau_{ij})_{,j} = 0 \quad \text{in} \quad \tau \tag{6}$$

$$(\delta\tau_{ij})\nu_j = 0 \quad \text{on} \quad \Sigma_T \tag{7}$$

$(\delta\tau_{ij})$ is arbitrary on Σ_u

$$W = \frac{1+\sigma}{2E}\tau_{ij}\tau_{ij} - \frac{\sigma}{2E}\theta^2 \quad ; \qquad \theta = \tau_{ij}$$

$$W' = \frac{1+\sigma}{2E}\tau'_{ij}\tau'_{ij} - \frac{\sigma}{2E}\theta'^2$$

$$= \frac{1+\sigma}{2E}\left(\tau_{ij} + \delta\tau_{ij}\right)\left(\tau_{ij} + \delta\tau_{ij}\right) - \frac{\sigma}{2E}\left(\theta + \delta\theta\right)^2$$

$$= \frac{1+\sigma}{2E}\tau_{ij}\tau_{ij} + \frac{1+\sigma}{2E}\tau_{ij}\delta\tau_{ij} + \frac{1+\sigma}{2E}\delta\tau_{ij}\delta\tau_{ij} - \frac{\sigma}{2E}\left(\theta^2 + 2\theta\delta\theta + (\delta\theta)^2\right)$$

$\Delta W = W' - W$ = Increase in strain energy density function.

$$\Delta W = \frac{1+\sigma}{E}\tau_{ij}\ \delta\tau_{ij} - \frac{\sigma}{E}\theta\,\delta\theta + W\ \left(\delta\tau_{ij}\right) \qquad \left(\because \delta\theta = \delta_{ij}\delta\tau_{ij}\right)$$

where $W\left(\delta\tau_{ij}\right) = \frac{1+\sigma}{2E}\left(\delta\tau_{ij}\right)\left(\delta\tau_{ij}\right) - \frac{\sigma}{2E}\left(\delta\theta\right)^2 \geq 0$

$$\Delta W = \left(\frac{1+\sigma}{E}\tau_{ij} - \frac{\sigma}{E}\theta\delta_{ij}\right)\left(\delta\tau_{ij}\right) + W\left(\delta\tau_{ij}\right)$$

$$= e_{ij}\left(\delta\tau_{ij}\right) + W\left(\delta\tau_{ij}\right)$$

$$= \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)\left(\delta\tau_{ij}\right) + W\left(\delta\tau_{ij}\right)$$

$$\Rightarrow \quad \Delta W = u_{i,j}\left(\delta\tau_{ij}\right) + W\left(\delta\tau_{ij}\right)$$

$$= \left(u_i\delta\tau_{ij}\right)_{,j} + W\left(\delta\tau_{ij}\right) - u_i\left(\delta\tau_{ij}\right)_{,j} \qquad (8)$$

$$\Delta U = \int_\tau \Delta W\, d\tau = \int_\tau \left(\left(u_i\delta\tau_{ij}\right)_{,j} + W\left(\delta\tau_{ij}\right) - u_i\left(\delta\tau_{ij}\right)_{,j}\right) d\tau$$

But $\left(\delta\tau_{ij}\right)_{,j} = 0$ in τ.

Therefore third part of integration vanishes.

Then using Gauss-divergence theorem,

$$\Delta U = \int_{\Sigma} \left(u_i \, \delta\tau_{ij}\right) \nu_j \, d\Sigma + \int_{\tau} W\left(\delta\tau_{ij}\right) d\tau$$

$$= \int_{\Sigma_u} \left(u_i \, \Delta T_i\right) d\Sigma + \int_{\tau} W\left(\delta\tau_{ij}\right) d\tau$$

$$\left(\because \Sigma = \Sigma_T + \Sigma_u, \quad \delta\tau_{ij}\nu_j = 0 \quad \text{on} \quad \Sigma_T \quad \text{But on} \quad \Sigma_u, \text{ we write} \quad \delta\tau_{ij}\nu_j = \Delta T_i\right)$$

Therefore

$$\Delta \left[U - \int_{\Sigma_u} \left(u_i \, T_i\right) d\Sigma \right] = \int_{\tau} W\left(\delta\tau_{ij}\right) d\tau \qquad (9)$$

We define V^* = complementary P. E. by

$$V^* = U - \int_{\Sigma_u} \left(u_i \, T_i\right) d\Sigma \qquad (10)$$

Then $\Delta V^* = \int_{\tau} W\left(\delta\tau_{ij}\right) d\tau$ (11)

But W > 0

$\Rightarrow \Delta V^* \geq 0$ (12)

$\Rightarrow V^*$ is minimum in τ.

Particular case:-

If the surface forces T_i are given over the entire surface, i.e., $\Sigma_u = 0$. Then $V^* = U$, i.e., complementary P.E. becomes strain energy. Then theorem of minimum complementary P.E. implies the Theorem of minimum strain energy (castigliano theorem).

9.7 Theorem of Minimum strain energy (Castigliano theorem)

Statement: The strain energy U of an elastic body in equilibrium under the action of prescribed surfaces forces is an absolute minimum on the set of all values of the functional U determined by the solution of the system

$$\tau_{ij,j} + F_i = 0 \quad \text{in} \quad \tau$$
$$\tau_{ij}\nu_j = T_i \quad \text{on} \quad \Sigma$$

Proof: Continuing from the previous theorem on complementary energy, we have

$$\left(\delta\tau_{ij}\right)\nu_j = 0 \quad \text{on} \quad \Sigma = \Sigma_T \cup \Sigma_u$$

and equation (9) reduces to

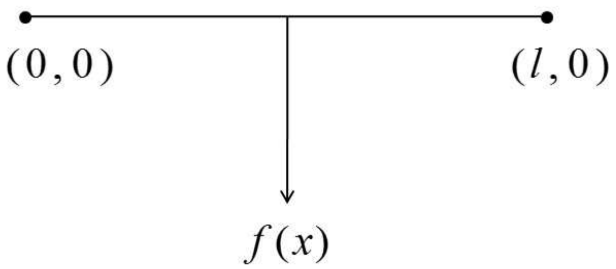
$$\Delta U = \int_\tau W\left(\delta\tau_{ij}\right)d\tau \geq 0$$

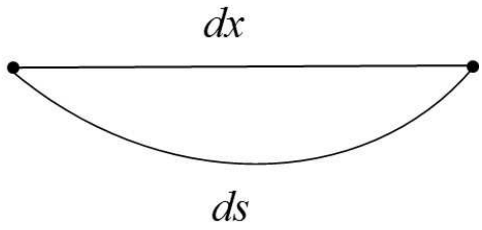
showing that the increment ΔU in the strain energy U of a body in equilibrium state is positive. Therefore, U is an absolute minimum.

Hence the result.

9.8 Deflection of an elastic string

Let a stretched string, with the end points fixed at (0, 0) and (*l*, 0). Let it be deflected by a transverse load f(x) per unit length of the string. We suppose that the transverse deflection y(x) is small (stretch in string is very small) and the change in the stretching farce T produced by the deflection is negligible.





These are the usual assumptions used in deriving equation for y(x) from consideration of static equilibrium. We deduce this equation from the principle of minimum P.E.

The Potential Energy, V is

$$V = U - \int_0^l f(x)\, y\, dx$$

where V is the gain in P.E.

U is the Strain energy

$\int_0^l f(x) y\, dx$ is the energy due to actual load.

where the strain energy U is equal to the product of the tensile force T by the total stretch *e* of the string.

Then, $e = \int_0^l (ds - dx) = \int_0^l \left(\sqrt{1 + y'^2} - 1\right) dx$ and we are dealing with the linear theory,

$y'^2 < 1$, and we have

$$e = \frac{1}{2}\int_0^l y'^2\, dx$$

$$\Rightarrow \quad U = \frac{T}{2}\int_0^l y'^2\, dx \qquad (\because\ U = Te)$$

$$\therefore \quad V = \int_0^l \left[\frac{T}{2} y'^2 - f(x)y\right] dx$$

Here $F = \left[\frac{T}{2} y'^2 - f(x)y\right]$

Euler's equation is

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

Therefore $\quad \frac{\partial F}{\partial y} = -f(x)\ ; \qquad \frac{\partial F}{\partial y'} = Ty'$

Then using this Euler's equation, we get

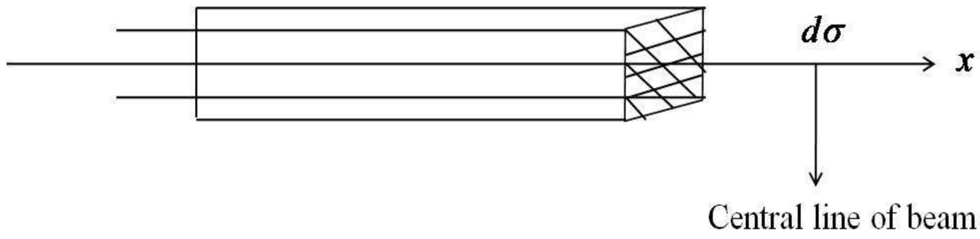
$$\frac{d}{dx}(Ty') + f(x) = 0$$

$$\Rightarrow Ty'' + f(x) = 0$$

This is required equation for the transverse deflection of the string under load f(x).

9.9 Deflection of central line of a beam

Let the axis of beam of constant cross-section coincide with the x-axis, and let that the beam is bent by a transverse load p = f(x) per unit of length of beam.



Here shearing stresses $\tau_{12}, \tau_{13}, \tau_{23}$ are negligible in comparison with the tensile stress.

$$\tau_{xx} = \frac{My}{I}$$

where M is Bending moment

y is Deflection

I is M. I.

The strain $e_{xx} = \frac{\tau_{xx}}{E} = \frac{My}{EI}$

where E is modulus of elasticity.

Strain energy density function W is

$$W = \frac{1}{2}\tau_{xx}\, e_{xx} = \frac{M^2 y^2}{2EI^2}$$

The strain energy per unit length of the beam

$$= \int_R W\, d\sigma = \frac{M^2}{2EI^2}\int_R y^2\, d\sigma$$

$$= \frac{M^2}{2EI^2} I = \frac{M^2}{2EI}$$

Also from Bernoulli-Euler Law,

$$M = -E\,I\,y''$$

$$\Rightarrow \int_R W\,d\sigma = \frac{1}{2}E\,I\,(y'')^2$$

The total strain energy U is obtained by integrating over the length of beam and we get

$$U = \int_0^l \frac{1}{2}E\,I\,(y'')^2\,dx$$

So

$$V = \int_0^l \frac{1}{2}EIy''^2\,dx - \int_0^l f(x)\,y\,dx$$

$$= \int_0^l \left[\frac{1}{2}EI\,y''^2 - f(x)y\right]dx$$

which is of the form

$$I(y) = \int_{x_1}^{x_2} F\left(x, y, y', y'', ---, y^{(n)}\right)dx$$

and Euler's equation is

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - .... = 0$$

Here

$$F = \frac{1}{2} E\,I\, y''^2 - f(x)y$$

$$F_y = -f(x)$$

$$F_{y'} = 0$$

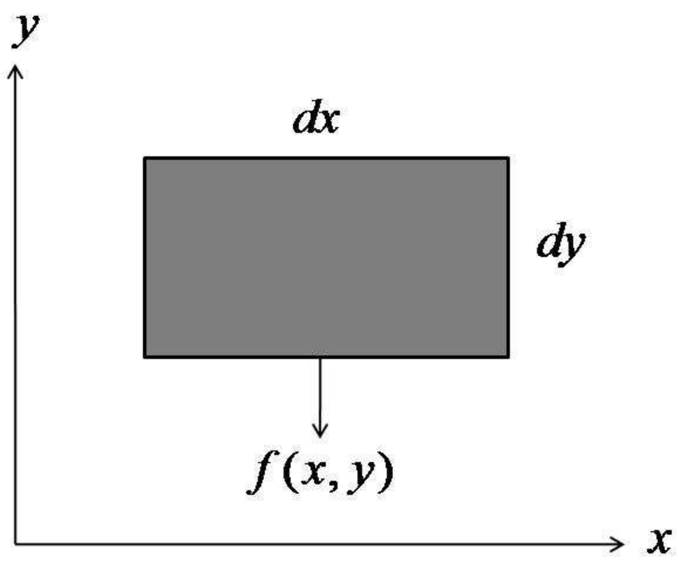
$$F_{y''} = EI\, y''$$

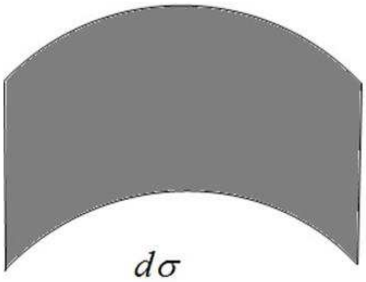
Then Euler’s equation becomes

$$\frac{d^2}{dx^2}(EIy'') - f(x) = 0$$

9.10 Deflection of an elastic membrane

Let membrane with fixed edges occupy some region in the xy-plane. We suppose that the membrane is stretched so that the tension T is uniform. Here load is f(x, y).





$d\sigma$

$$e = \iint_R \left(d\sigma - dx\,dy\right)$$

$$e = \iint_R \left(\sqrt{u_x^2 + u_y^2 + 1} - 1\right) dx\,dy$$

where $d\sigma = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + 1}$ is the element of area of membrane in deformed state. If the displacement u and its first derivatives are small, then

$$e = \frac{1}{2}\iint_R \left(u_x^2 + u_y^2\right) dx\,dy$$

Hence $U = \dfrac{T}{2}\displaystyle\iint_R \left(u_x^2 + u_y^2\right) dx\,dy$

and $V = \displaystyle\iint_R \left[\frac{T}{2}\left(u_x^2 + u_y^2\right) - f(x,y)\,u\right] dx\,dy$

where u is the deflection $u(x,y)$

Here $F = \dfrac{T}{2}\left(u_x^2 + u_y^2\right) - f(x,y)u$

$$\frac{\partial F}{\partial u} = -f(x,y)\,, \qquad \frac{\partial F}{\partial u_x} = Tu_x\,, \qquad \frac{\partial F}{\partial u_y} = Tu_y$$

Then the Euler's equation,

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial u_y}\right) = 0$$

$$\Rightarrow \qquad -f(x,y) - \frac{\partial T u_x}{\partial x} - \frac{\partial T u_y}{\partial y} = 0$$

$$\Rightarrow \qquad T u_{xx} + T u_{yy} + f(x,y) = 0$$

$$\Rightarrow \qquad T\nabla^2 u + + f(x,y) = 0$$

which is required equation.

9.11 Summary

We have studied about Variational methods and some theorems namely, Theorems of minimum potential energy, Theorems of minimum complementary energy, Theorems of minimum strain energy, Reciprocal theorem of Betti and Rayleigh. We have also discussed about Deflection of elastic string, elastic beam and elastic membrane.

9.12 Keywords: Potential energy, Complementary energy, Strain energy, Betti and Rayleigh, Deflection, elastic string, Elastic membrane

9.13 Self-assessment Questions

Q 1. State and prove Theorems of minimum potential energy.

Q 2. State and prove Betti's reciprocal theorem.

Q 3. State and prove Theorems of minimum Complementary energy.

Q 4. Discuss the problem of Deflection of central line of a beam by transverse load.

Q 5. Discuss the problem of Deflection of an elastic string by transverse load and hence give its Euler's equation.

Q 6. Discuss the problem of Deflection of an elastic membrane by transverse load f(x, y).

9.14 Suggested Readings

1. I.S. Sokolnikoff, Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company Ltd., New Delhi.
2. Y.C. Fung, Foundations of Solid Mechanics, Prentice Hall, New Delhi.
3. S. Timoshenko and N. Goodier, Theory of Elasticity, McGraw Hill, New York.
4. Martin H. Sadd., Elasticity Theory, Applications and Numerics AP (Elsevier).
5. A.E.H. Love, A Treatise on the Mathematical Theory of Elasticity, 4^{th} Ed., Dover Publications, New York.

Chapter 10

Direct methods

10.1 Objectives

In this chapter, we shall discuss how to find solution of Euler's equation in the calculus of variations by direct methods namely Ritz method, Galerkin method and Kantorovich method. Some numerical examples based on these methods are also given.

10.2 Introduction

It was demonstrated in Sections (9.4) and (9.6) that the determination of functions that minimize the functional (equation (5) in section 9.4) for the potential energy V, or the expression (equation (10) in section 9.6) for the complementary energy V^* , is equivalent to obtaining solutions of appropriate Euler's equations. In the variational problem V = min, the Euler equations are the Cauchy equilibrium equations, while, in the problem V^* = min, they are the compatibility equations.

In the previous chapter, we have studied some uses of minimum principles in the derivation of the differential equations for specific problems. However, so by far more important use of these principles relates to the construction, with the aid of direct methods of calculus of variations, of sequences of functions which converge to desired solutions of Euler's equations. One such direct method was proposed by Lord Rayleigh and, independently and from a more general point of view, by W. Ritz. The other direct methods in the calculus of variations were proposed by R. Courant, K. Friedrichs, B. G. Galerkin, L. V. Kantorovich, S. G. Mikhlin, E. Trefftz, and others.

10.3 Rayleigh –Ritz Method (or Ritz's Method in one dimension)

How to find approximate solution of variational problem using Ritz's Method.

Consider the variational problem

$$I[y] = \int_{x_1}^{x_2} F(x, y, y')\, dx = \min \tag{1}$$

in which all admissible function y = y (x) are such that

$$y(x_1) = y_1\,, \qquad y(x_2) = y_2 \tag{2}$$

We know that such a function y is a solution of the Euler's equation

$$F_y - \frac{d}{dx} F'_y = 0 \tag{3}$$

A direct method to obtain the desired function was proposed by W. Ritz in 1911.

In this method, we construct a sequence of functions which converge to desired solution of the Euler's equation (3).

Outlines of the Ritz Method:

Let y = y* (x) be the exact solution of the given variational problem. Let I(y*) = m be the minimum value of the functional in (1).

In this method, one tries to find a sequence { $\bar{y}_n(x)$ } of admissible functions such that

$$\lim_{n \to \infty} I(\bar{y}_n(x)) = m \tag{4}$$

so that

$$\lim_{n\to\infty} \bar{y}_n(x) = y^*(x) \tag{5}$$

is the required function.

According to Ritz, solution of (1) can be approximated by a linear combination of suitable chosen co-ordinate functions $\{\phi_i(x)\}$.

Let approx. solution is taken as

$$y_n(x) = \phi_0 + c_1\phi_1(x) + c_2\phi_2(x) + ... + c_n\phi_n(x) \tag{6}$$

where c_1, c_2 ………, c_n are constants to be determined and n is the no. of parameters. The functions $\{\phi_i(x)\}$ are to be so chosen that the (6) satisfies the given B.C.'s. Generally, we chosen ϕ_0 so that it takes on prescribed values at the ends and remaining $\phi_j(x)$, $(j \geq 1)$ vanish at both ends.

The approx. solution (6) is then put in (1) and required integration is performed, getting a function of parameters c_i's ,

$$I = I(c_1, c_2, ..., c_n) \tag{7}$$

which can be minimized on using differential calculus, i.e., by solving n equations

$$\frac{\partial I}{\partial c_i} = 0 \tag{8}$$

If $\bar{c}_i\,(i = 1,2,......, n)$ are the n parameters obtained by solving (8), then approx. minimizing function is

$$\bar{y}_n(x) = \phi_0 + \bar{c}_1\phi_1(x) + \bar{c}_2\phi_2(x) + ... + \bar{c}_n\phi_n(x) .$$

Example: - Apply Ritz's method to solve the problem

$$I[y] = \int_0^1 \left(y'^2 - y^2 - 2xy \right) dx = \min \tag{1}$$

$$y(0) = y(1) = 0\,. \tag{2}$$

Solution: - we choose ϕ_0 =0

$$\phi_k = x^k (1-x) \qquad \text{(Choose so that at end points, } \phi_k \text{ to vanish)} \tag{3}$$

i.e., approx. solution is

$$\begin{aligned} y_n(x) &= x(1-x)c_1 + c_2 x^2 (1-x) + \dots\dots\dots + c_n x^n (1-x) \\ &= x(1-x)\left\{ c_1 + c_2 x + \dots\dots\dots + c_n x^{n-1} \right\} \end{aligned} \tag{4}$$

Take n = 1, so that approx. solution is

$$\begin{aligned} y_1(x) &= x(1-x)c_1 \\ y_1'(x) &= (1-2x)c_1 \end{aligned} \tag{5}$$

Put y_1 in place of y in (1), we get

$$\int_0^1 \left((1-2x)^2 c_1^2 - x^2 (1-x)^2 c_1^2 - 2x^2 (1-x) c_1 \right) dx = \min$$

$$\Rightarrow c_1^2 \left(x + \frac{3x^3}{3} - \frac{4x^2}{2} - \frac{x^5}{5} + \frac{2x^4}{4} \right)_0^1 - c_1 \left(\frac{2x^3}{3} - \frac{2x^4}{4} \right)_0^1 = \min$$

$$\Rightarrow \frac{3}{10} c_1^2 - \frac{1}{6} c_1 = \min$$

So $I(c_1) = \dfrac{3}{10} c_1^2 - \dfrac{1}{6} c_1$

The parameter c_1 is determined from

$$\frac{\partial I(c_1)}{\partial c_1} = 0$$

$$\Rightarrow \quad \frac{\partial}{\partial c_1}\left(\frac{3}{10}c_1^2 - \frac{1}{6}c_1\right) = 0$$

$$\Rightarrow \quad \frac{3}{5}c_1 - \frac{1}{6} = 0 \qquad \Rightarrow c_1 = \frac{5}{18}$$

Therefore, the Ist approx. solution is $\bar{y}_1 = \frac{5}{18}x(1-x)$

Take n = 2, so that approx. solution is

$$y_2(x) = x(1-x)(c_1 + c_2 x)$$
$$y_2'(x) = \left[(1-2x)c_1 + c_2\left(2x - 3x^2\right)\right]$$

Put in (1) in place of y(x), we get

$$\int_0^1 \left(\begin{array}{l} (1-4x+3x^2+2x^3-x^4)c_1^2 + c_2^2(4x^2-12x^3+8x^4+2x^5-x^6) \\ + c_1c_2\left(4x-14x^2+10x^3+4x^4-2x^5\right) + c_1\left(-2x^2+2x^3\right) + c_2\left(-2x^3+2x^4\right) \end{array} \right) dx = \min$$

$$I(c_1, c_2) = \frac{3}{10}c_1^2 + \frac{13}{105}c_2^2 + \frac{3}{10}c_1c_2 - \frac{1}{6}c_1 - \frac{1}{10}c_2 = \min$$

Now c_1 and c_2 are determined by putting

$$\frac{\partial I}{\partial c_1} = 0, \qquad \frac{\partial I}{\partial c_2} = 0$$

So we have

$$\frac{3}{5}c_1 + \frac{3}{10}c_2 = \frac{1}{6},$$

and $\frac{26}{105}c_2 + \frac{3}{10}c_1 = \frac{1}{10}$

$$\Rightarrow \qquad c_1 = \frac{71}{369}, \quad c_2 = \frac{7}{41}$$

Then 2nd approx. solution is

$$y_2(x) = \bar{y}_2(x) = x(1-x)\left(\frac{71}{369} + \frac{7}{41}x\right)$$

Example: - Apply Ritz's method to solve the problem

$$I[y] = \int_0^1 \left(y'^2 + y^2\right)dx = \min \qquad (1)$$

$$y(0) = y(1) = 0\,. \qquad (2)$$

Solution: we choose $\phi_0 = x$ and

$\phi_k = x^k(1-x)$, k = 1,2,……,n.

10.4 Ritz's Method in two dimension

Consider the functional in the form

$$I\left[u(x,y)\right] = \iint_R F(x,y,u,u_x,u_y)\,dx\,dy = \min \qquad (1)$$

Let approx. solution is

$$u_n(x,y) = \sum_{j=1}^{n} c_j \phi_j(x,y)$$

so that $\phi_j(x,y)$ are to satisfy the given boundary conditions.

i.e., approximate solution is

$$u_n = (c_1 + c_2 + \ldots\ldots.. + c_n)\,\phi(x,y)$$

where $c_1, c_2, ..., c_n$ are the parameters. Use in (1) and perform integration, then

put $\dfrac{\partial I}{\partial c_i} = 0$

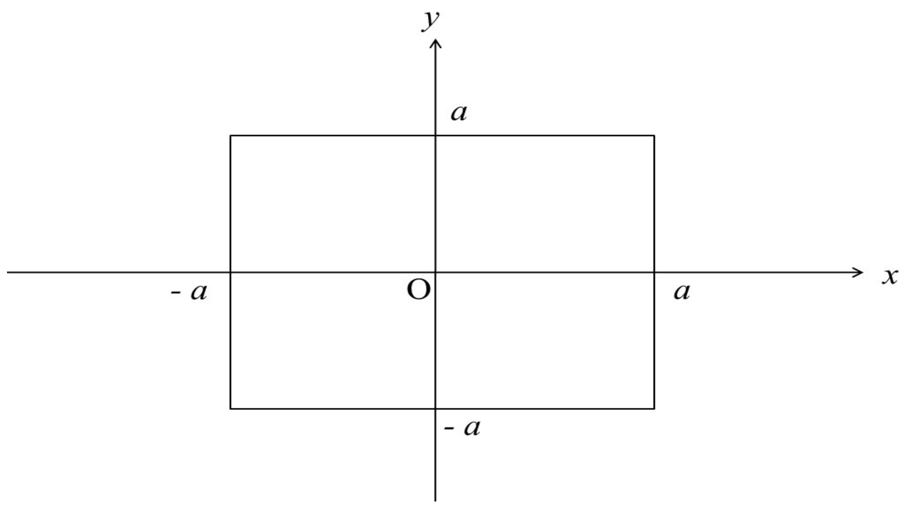
If $\overline{c_i}$ are values obtained, then approx. Minimizing solution is

$$\overline{u_n} = \left(\overline{c_1} + \overline{c_2} + ... + \overline{c_n}\right) \phi(x, y)$$

Example:- $I[u(x, y)] = \iint\limits_R \left[\left(\dfrac{\partial u}{\partial x} \right)^2 + \left(\dfrac{\partial u}{\partial y} \right)^2 - 2u \right] dx\, dy = \min.$

where R is square $|x| \le a,\ |y| \le a$ and u = 0 on boundary of R.

Solution: - Let R is square $|x| \le a,\ |y| \le a$ (shown in figure)



Let n = 1,

Assume approx. solution as

$$u \cong u_1(x, y) = c_1\left(x^2 - a^2\right)\left(y^2 - a^2\right)$$

where c_1 is the parameter to be determined.

Then we see that $u_1(x, y)$ clearly satisfies the given boundary conditions.

$$\frac{\partial u}{\partial x} = u_x = c_1 2x\left(y^2 - a^2\right)$$
$$\frac{\partial u}{\partial y} = u_y = c_1 2y\left(x^2 - a^2\right)$$

Then putting in (1), we get

$$I[u_1] = \int_{-a}^{a}\int_{-a}^{a}\left[\left(c_1 2x\left(y^2 - a^2\right)\right)^2 + \left(c_1 2y\left(x^2 - a^2\right)\right)^2 - 2c_1\left(x^2 - a^2\right)\left(y^2 - a^2\right)\right] dy\,dx$$

$$I[u_1] = \int_{-a}^{a}\int_{0}^{a}\left[c_1^2 4x^2\left(y^2 - a^2\right)^2 + c_1^2 4y^2\left(x^2 - a^2\right)^2 - 2c_1\left(x^2 - a^2\right)\left(y^2 - a^2\right)\right] dy\,dx$$

$$= 2\int_{-a}^{a}\int_{0}^{a}\left[4c_1^2 x^2\left(y^4 + a^4 - 2a^2y^2\right) + 4c_1^2 y^2\left(x^2 - a^2\right)^2 - 2c_1\left(x^2 - a^2\right)\left(y^2 - a^2\right)\right] dy\,dx$$

$$= 2\int_{-a}^{a}\left[4c_1^2 x^2\left(\frac{y^5}{5} + a^4 y - 2a^2\frac{y^3}{3}\right) + 4c_1^2\frac{y^3}{3}\left(x^2 - a^2\right)^2 - 2c_1\left(x^2 - a^2\right)\left(\frac{y^3}{3} - a^2 y\right)\right]_0^a dx$$

$$= 2\int_{-a}^{a}\left[4c_1^2 x^2\left(\frac{a^5}{5} + a^5 - \frac{2}{3}a^5\right) + 4c_1^2\frac{a^3}{3}\left(x^2 - a^2\right)^2 - 2c_1\left(x^2 - a^2\right)\left(\frac{a^3}{3} - a^3\right)\right]_0^a dx$$

$$= 2\int_{-a}^{a}\left[4c_1^2 x^2\left(\frac{3+15-10}{15}\right)a^5 + 4c_1^2\frac{a^3}{3}\left(x^2 - a^2\right)^2 + \frac{4}{3}c_1 a^3\left(x^2 - a^2\right)\right] dx$$

$$= 4\int_{0}^{a}\left[4c_1^2 x^2.\frac{8}{15}a^5 + \frac{4}{3}c_1^2 a^3\left(x^2 - a^2\right)^2 + \frac{4}{3}c_1^2 a^3\left(x^2 - a^2\right)\right] dx$$

$$= \frac{256}{45}c_1^2 a^8 - \frac{32}{9}c_1 a^6$$

Thus $I[c_1] = \frac{256}{45} a^8 c_1^2 - \frac{32}{9} a^6 c_1$

Then c_1 is given by

$$\frac{\partial I(c_1)}{\partial c_1} = 0$$

$$\Rightarrow \quad \frac{512}{45} a^8 c_1 - \frac{32}{9} a^6 = 0$$

$$\Rightarrow \quad c_1 = \frac{5}{16a^2}$$

Therefore, Ist approx. solution is

$$u \cong u_1(x, y) = \frac{5}{16a^2}\left(x^2 - a^2\right)\left(y^2 - a^2\right)$$

10.5 Galerkin Method

In 1915, Galerkin proposed a method of finding an approximate solution of the boundary value problems in mathematical physics. This method shall have wider scope than the method of Ritz.

Here approx. solution of Boundary Value Problem can be obtained.

Let us consider linear differential equation

L[u] = 0 in R (1)

subjected to some linear homogenous boundary conditions.

It is assumed, for the sake of simplicity that the domain R is two-dimensional.

We take an approx. solution of the problem in the form

$$u_n(x, y) = \sum_{j=1}^{n} a_j \phi_j(x, y) \quad (2)$$

where $\phi_j(x, y)$ are suitable co-ordinate functions and a_j are constant.

We suppose that the functions $\phi_j(x, y)$ satisfy the same boundary conditions as the exact solution u(x, y) and that the set { ϕ_j} is complete in the sense that every piecewise continues function $f(x, y)$, say, can be approximated in R by the sum $\sum_{j=1}^{N} a_j \phi_j(x, y)$ in such a way that

$$\delta_N \equiv \iint_R \left(f - \sum_{j=1}^{N} c_j \phi_j \right)^2 dx\, dy \qquad (3)$$

can be made as small as we wish.

Ordinarily, the finite sum u_n given in (2) will not satisfy (1) and the substitution of u_n will yield

$$L(u_n) = \varepsilon_n(x, y) \ ; \quad \varepsilon_n(x, y) \neq 0 \quad \text{in} \quad R \qquad (4)$$

If maximum of $\varepsilon_n(x, y)$ is small, we can consider $u_n(x, y)$given is (2) as a satisfactory approximation to the exact solution u(x, y).

Thus, to get a good approximation, we have to choose the constants a_j so as to minimize the error function $\varepsilon_n(x, y)$.

A reasonable minimization technique is suggested by the following:

Galerkin established that If one represents u(x, y) by the series $u\ (x,y)=\sum_{i=1}^{\infty} a_i\ \phi_i(x,y)$, with suitable properties and consider the nth partial sum $u_n\ (x,y)=\sum_{i=1}^{n} a_i\ \phi_i(x,y)$, then the orthogonality condition

$$\iint_R L[u_n]\phi_i(x,y)\,dx\,dy=0 \quad \text{as} \quad n\to\infty \tag{5}$$

is equivalent to the statement L[u] = 0 (6)

This led Galerkin to impose on the function $L(u_n)$ a set of orthogonality conditions (now called Galerkin conditions)

$$\iint_R L\ [u_n]\phi_i\ (x,y)\ dx\,dy=0 \quad ; \quad (i=1,\ 2,.........,n) \tag{7}$$

This yields the set of equations

$$\iint_R L\left(\sum_{j=1}^{n} a_j\ \phi_j\right)\phi_i\ \ dx\,dy=0 \quad ; \quad (i=1,\ 2,.........,n) \tag{8}$$

This set of n equations in (8) determines the constants a_j in the approximate solution (2).

Remark 1. When the differential equation and the boundary conditions are self-adjoint and the corresponding functional I (u) in the problem

I (u) = min , (9)

is positive definite, then the system of Galerkin equation in (8) is equivalent to the Ritz system

$$\frac{\partial}{\partial a_j} I(u_n) = 0 \qquad (10)$$

Remark 2. It is important to the note that in Galerkin's formulation, there is no reference to any connection of equation (1) with a variational problem. Indeed, the Galerkin method can be applied to a wider class of problems phrased in terms of integrals and other types of functional equations.

Example: Use Galerkin Method to find approx. solution of

$$\nabla^2 \psi = -2 \qquad \text{in R,} \qquad (1)$$

$$\psi = 0 \quad \text{on boundary C of R,} \qquad (2)$$

where R is the rectangle, $|x| \le a$, $|y| \le b$.

Solution: Let R is the rectangle, $|x| \le a$, $|y| \le b$ (shown in figure)

Now we have to solve the system consisting of equations (1) and (2) by using the Galerkin method. We write (1) as

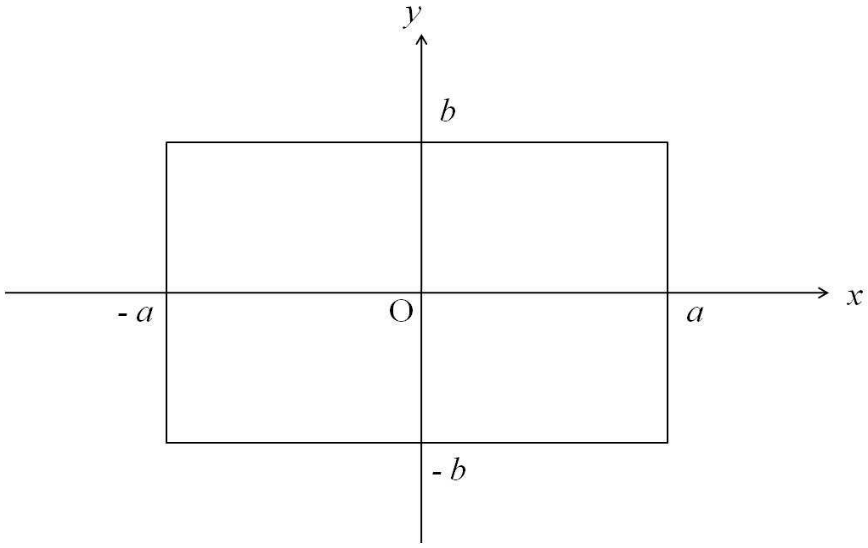
$$L(\psi) = 0\,, \qquad (3)$$

where

$$L = \nabla^2 + 2 \qquad (4)$$

We take an approximate solution in the form

$$\psi_n(x, y) = \left(x^2 - a^2\right)\left(y^2 - b^2\right)\left(a_1 + a_2 x^2 + a_3 y^2 + \ldots\ldots.. + a_n x^{2k} y^{2k}\right) \qquad (5)$$



This approximate solution satisfies the boundary conditions in (3). Here

$a_1, a_2, a_3, ..., a_n$ are constants to be determined by using Galerkin method.

Let n =1, Ist approx. solution is

$$\psi_1(x, y) = a_1\phi_1 = a_1\left(x^2 - a^2\right)\left(y^2 - b^2\right) \tag{6}$$

with

$$\phi_1 = \left(x^2 - a^2\right)\left(y^2 - b^2\right) \tag{7}$$

Then ψ_1 satisfies given B.C.

Following Galerkin, a_1 is determined by orthogonality condition,

$$\int_{-a}^{a}\int_{-b}^{b}\left[\left(\nabla^2\psi_1 + 2\right)\phi_1\right] dy\ dx = 0$$

Or

$$\int_{-a}^{a}\int_{-b}^{b}\left[\left(\nabla^2\psi_1 + 2\right)\left(x^2 - a^2\right)\left(y^2 - b^2\right)\right] dy\ dx = 0 \tag{8}$$

where $\nabla^2\psi_1 = \dfrac{\partial^2\psi_1}{\partial x^2} + \dfrac{\partial^2\psi_1}{\partial y^2}$

Now we have,

$$\begin{aligned}\nabla^2\psi_1 = \frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} &= a_1\left[2\left(x^2 - a^2\right) + 2\left(y^2 - b^2\right)\right] \\ &= 2a_1\left[\left(x^2 - a^2\right) + \left(y^2 - b^2\right)\right]\end{aligned} \qquad (9)$$

$$\therefore \quad \int_{-a}^{a}\int_{-b}^{b} \left[\nabla^2\psi_1 + 2\right]\left(x^2 - a^2\right)\left(y^2 - b^2\right) dy\, dx = 0 \qquad \text{[Using (8)]}$$

$$\Rightarrow \quad \int_{-a}^{a}\int_{-b}^{b} \left[2a_1\left\{\left(x^2 - a^2\right) + \left(y^2 - b^2\right) + 2\right\}\right]\left(x^2 - a^2\right)\left(y^2 - b^2\right) dy\, dx = 0 \qquad \text{[Using (9)]}$$

$$\Rightarrow \quad 2\int_{-a}^{a}\int_{-b}^{b} \left[a_1\left(x^2 - a^2\right)^2\left(y^2 - b^2\right) + a_1\left(y^2 - b^2\right)^2\left(x^2 - a^2\right) + \left(x^2 - a^2\right)\left(y^2 - b^2\right)\right] dy\ dx = 0$$

$$\Rightarrow \quad 8\int_{0}^{a}\int_{0}^{b} \left[a_1\left(x^2 - a^2\right)^2\left(y^2 - b^2\right) + a_1\left(y^2 - b^2\right)^2\left(x^2 - a^2\right) + \left(x^2 - a^2\right)\left(y^2 - b^2\right)\right] dy\ dx = 0$$

$$\Rightarrow \quad 8\int_{0}^{a}\int_{0}^{b} \left(x^2 - a^2\right)\left[\left(y^2 - b^2\right)\left\{a_1\left(x^2 - a^2\right) + 1\right\} + a_1\left(y^4 + b^4 - 2y^2b^2\right)\right] dy\ dx = 0$$

On integration w. r. t. y and x, we get

$$\frac{128}{45}a^3b^3\left(a^2 + b^2\right)a_1 - \frac{32}{9}a^3b^3 = 0$$

$$\Rightarrow a_1 = \frac{5}{4\left(a^2 + b^2\right)}$$

Then Ist approx. solution is

$$\psi_1(x, y) = \frac{5}{4\left(a^2 + b^2\right)}\left(x^2 - a^2\right)\left(y^2 - b^2\right)$$

10.6 Kantrovich Method

In 1932, Kantorvich proposed a generalization of the Ritz method. The essence of the method consists in the reduction of integration of partial differential equations (Euler's equation) to the integration of systems of ordinary differential equations.

It is applied to variational problems that involve several independent variables.

In the application of the Ritz method to the problem

$$I[u(x,y)] = \iint_R F(x, y, u, u_x, u_y)\, dx\, dy = \min \tag{1}$$

where u(x, y) takes on given values at boundary of region R.

Approx. solution of variational problem (1) can be considered in the form

$$u_n(x,y) = \sum_{j=1}^{n} c_j(x)\ \phi_j(x,y) \tag{2}$$

where $c_j(x)$ are unknown functions of the independent variable x, and $\phi_j(x,y)$ are suitable chosen coordinate functions so as to satisfy the same boundary conditions as imposed on u.

We then determined the coefficients $c_j(x)$ so as to minimize $I(u_n)$.

We put (2) in (1) in place u(x, y),

$$I(u_n) = I\left[\sum_{j=1}^{n} c_j(x)\ \phi_j(x,y)\right] = \min \tag{3}$$

Since $\phi_j(x,y)$ are known functions, we perform integration w.r.t. y and we get reduced form of the problem (1) as

$$I[u_n] = \int_{x_1}^{x_2} G\left(x, c_j(x), c'_j(x)\right) dx = \min \qquad (4)$$

Kantorvich proposed to determine the function $c_j(x)$ so that they minimize the functional (4).

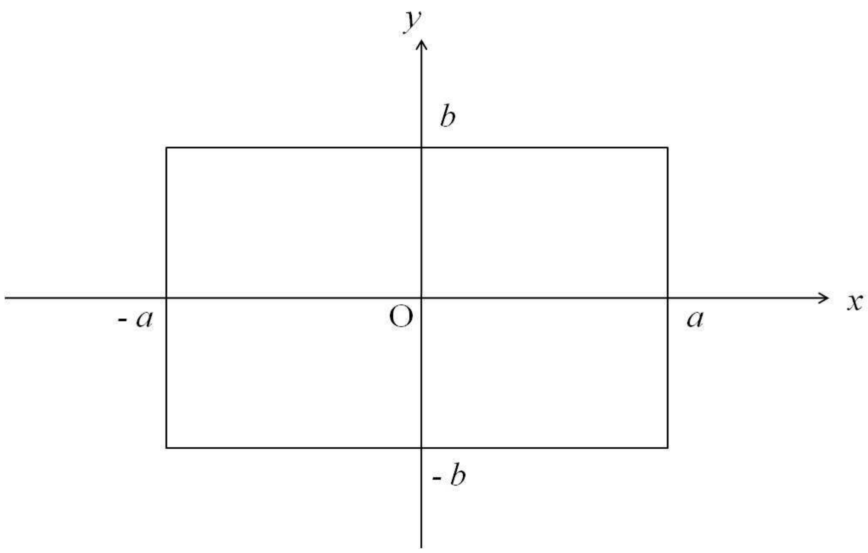
The functions $c_j(x)$ are then determined by solving Euler's equation corresponding to (4), which is a 2nd order differential equation.

Example: - $\nabla^2 u = -1$ (1)

in the rectangle $-a \le x \le a, \quad -b \le y \le b$ (2)

where u = 0 on the boundary.

Solution: - Let R is the rectangle, $|x| \le a$, $|y| \le b$ (shown in figure)



Equation (1) is Euler's equation for the functional

$$I[u(x,y)] = \iint_R \left[u_x{}^2 + u_y{}^2 - 2u \right] dx\, dy = \min. \tag{3}$$

Let approx. solution is

$$u_1(x,y) = c_1(x)\,(b^2 - y^2) \tag{4}$$

Then u_1 satisfied B.C. on y = $\pm b$.

Substitute (4) in place of u(x, y) in (3), we get

$$I[u_1] = \int_{-a}^{a} \int_{-b}^{b} \left[\left(c_1'(b^2 - y^2)\right)^2 + (c_1 2y)^2 - 2c_1(b^2 - y^2) \right] dy\, dx \tag{5}$$

Perform integration w.r.t. y, we get

$$I[u_1] \;\; = 2 \int_{-a}^{a} \int_{0}^{b} \left[c_1'^2 (b^4 + y^4 - 2b^2y^2) + 4c_1^2 y^2 - 2c_1(b^2 - y^2) \right] dy\, dx$$

$$= 2\int_{-a}^{a} \left\{ \left(b^4 y + \frac{y^5}{5} - 2\frac{b^2y^3}{3} \right) c_1'^2 + 4c_1^2 \frac{y^3}{3} - \left(b^2 y - \frac{y^3}{3} \right) \right\}_0^b dx$$

$$= 2\int_{-a}^{a} \left[\left(b^5 + \frac{b^5}{5} - \frac{2}{3}b^5 \right) c_1'^2 + \frac{4}{3}c_1^2 b^3 - 2c_1\left(b^3 - \frac{b^3}{3} \right) \right] dx$$

$$= 2\int_{-a}^{a} \left[c_1'^2 \frac{8b^5}{15} + \frac{4}{3}c_1^2 b^3 - \frac{4c_1}{3} b^3 \right] dx$$

$$\Rightarrow I[c_1] = \int_{-a}^{a} \left(\frac{16}{15} b^5 c_1'^2 + \frac{8}{3} b^3 c_1^2 - \frac{8}{3} b^3 c_1 \right) dx \tag{6}$$

$c_1(x)$ is determined by solving Euler's eq. corresponding to (6).

The general Euler's equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Here y is $c_1(x)$; $c'_1 = c'_1(x)$;

y' is $c_1'(x)$

Then the Euler's equation becomes,

$$\frac{d}{dx}\left[\frac{32}{15} b^5 c_1'(x)\right] - \frac{16}{3} b^3 c_1(x) + \frac{8}{3} b^3 = 0$$

$$\Rightarrow \frac{32}{15} b^5 c_1''(x) - \frac{16}{3} b^3 c_1(x) + \frac{8}{3} b^3 = 0$$

Or $c_1'' - \frac{5}{2b^2} c_1 = -\frac{5}{4b^2}$ (7)

which is homogeneous linear D.E. with constant coefficients.

Characteristics equation is

$$m^2 - \frac{5}{2b^2} = 0$$

$$\Rightarrow m = \pm \sqrt{\frac{5}{2}} \times \frac{1}{b}$$

C.F. = $Ae^{\sqrt{\frac{5}{2}} \times \frac{1}{b} x} + Be^{-\sqrt{\frac{5}{2}} \times \frac{1}{b} x}$

P.I. = $\frac{1}{D^2 - \frac{5}{2b^2}}\left(\frac{-5}{4b^2}\right) e^0 = \frac{1}{2}$

Then solution of equation (7) is

$c_1(x) = Ae^{\sqrt{\frac{5}{2}} \times \frac{x}{b}} + Be^{-\sqrt{\frac{5}{2}} \times \frac{x}{b}} + \frac{1}{2}$ (8)

where A and B are the arbitrary constants to be determined from B.C.

As $c_1(a) = c_1(-a) = 0$

Now

$$c_1(a) = 0 \qquad \Rightarrow \quad Ae^{\sqrt{\frac{5}{2}}\times\frac{a}{b}} + Be^{-\sqrt{\frac{5}{2}}\times\frac{a}{b}} = -\frac{1}{2} \qquad (9)$$

and $c_1(-a) = 0 \qquad \Rightarrow \quad Ae^{-\sqrt{\frac{5}{2}}\times\frac{a}{b}} + Be^{\sqrt{\frac{5}{2}}\times\frac{a}{b}} = -\frac{1}{2}$

(10)

Solving (9) and (10), we get

$$\Rightarrow \qquad A = B = -\frac{1}{2}\frac{e^{\sqrt{\frac{5}{2}}\times\frac{a}{b}}}{1+\left(e^{\sqrt{\frac{5}{2}}\times\frac{a}{b}}\right)^2}$$

(11)

Substituting the values of A and B from equation (11) into (8), we get

$$c_1(x) = A\left(e^{\sqrt{\frac{5}{2}}\times\frac{x}{b}} + e^{-\sqrt{\frac{5}{2}}\times\frac{x}{b}}\right) + \frac{1}{2}$$

$$= \frac{1}{2}\left[1 - \frac{\left(e^{\sqrt{\frac{5}{2}}\times\frac{x}{b}} + e^{-\sqrt{\frac{5}{2}}\times\frac{x}{b}}\right)}{\left(e^{\sqrt{\frac{5}{2}}\times\frac{a}{b}} + e^{-\sqrt{\frac{5}{2}}\times\frac{a}{b}}\right)}\right]$$

So $u_1(x,y)=\frac{1}{2}\left(y^2-b^2\right)\left[\frac{\cosh\left(\sqrt{\frac{5}{2}}\times\frac{x}{b}\right)}{\cosh\left(\sqrt{\frac{5}{2}}\times\frac{a}{b}\right)}-1\right]$

10.7 Summary

We have find solutions of Euler's equation by direct methods such as Ritz method, Galerkin and Kantorovich methods.

10.8 Keywords: Direct methods, Ritz method, Galerkin method, Kantorovich method, Euler's equation

10.9 Self-assessment Questions

Q 1. Use Kantorovich method to find an approximate solution of the Poisson's equation

$\nabla^2 u = -1,$

in the square $-a \le x \le a, \quad -a \le y \le a$,

where u = 0 on the boundary.

Q 2. Apply Ritz's method to solve the problem

$$I[y]=\int_0^1\left(y'^2-y^2-2xy\right)dx=\min$$

$y(0)=y(1)=0$,

by considering the approximate solution in the form $y = x(1-x)a_1$

Q 3. Apply Ritz's method to solve the problem

$$I[u(x,y)] = \iint_R \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - 2u \right] dx\, dy = \min .$$

where R is square $|x| \le a,\ |y| \le a$ and u = 0 on boundary of R.

Q 4. Use Galerkin Method to find approx. solution of

$\nabla^2 \psi = -2$ in R.

$\psi = 0$ on boundary of R,

where R is the square, $|x| \le a$, $|y| \le a$.

10.10 Suggested Readings

1. I.S. Sokolnikoff, Mathematical Theory of Elasticity, Tata McGraw Hill Publishing Company Ltd., New Delhi.
2. Y.C. Fung, Foundations of Solid Mechanics, Prentice Hall, New Delhi.
3. S. Timoshenko and N. Goodier, Theory of Elasticity, McGraw Hill, New York.
4. Martin H. Sadd., Elasticity Theory, Applications and Numerics AP (Elsevier).